

§30. The Countability Axioms.

Def. X is 1st countable if it has a countable basis at each point.

X is 2nd countable if it has a countable basis.

Rmk. 2nd ctbl implies 1st ctbl.

• Metric spaces are 1st ctbl.

Thm. A subspace/countable product of 1st/2nd countable spaces is 1st/2nd ctbl.

ex. $\mathbb{R}$ and $(\mathbb{R}^{\omega}, \text{product})$ are 2nd ctbl.

$(\mathbb{R}^{\omega}, \text{uniform})$ is not 2nd ctbl, b/c it has an uncountable discrete subspace $\{0,1\}^{\omega}$.

Other notions of countability:

Thm Let X be 2nd ctbl. Then

(1) X is Lindelöf, i.e., every open cover of X has a countable subcover.

(2) X is separable, i.e., there exists a countable dense subset A of X .
($\uparrow$ means $\bar{A} = X$)

Rmk The converse holds for metric spaces and 1st ctbl top'l gps, i.e., if X is either metrizable or is a 1st ctbl top'l gp, and X is either Lindelöf or separable, then X is 2nd ctbl.

Ex. $\mathbb{R}_\ell$ is 1st ctbl, Lindelöf, and separable, but not 2nd ctbl.

$\uparrow$ $\mathbb{R}$ w/ lower limit topology.

(See Example 3 on p. 192 for a pt)

Ex. "Sorgenfrey plane" $\mathbb{R}_\ell^2$ is not Lindelöf $\Rightarrow$ Lindelöfness not closed under finite products or taking subspaces.
(Example 4 on p. 193)

Ex. Ordered square I_0^2 is compact (and, a fortiori, Lindelöf) $\Rightarrow$ subspaces.
But the subspace $I_x(0,1)$ is not Lindelöf.

§31. The Separation Axioms

Def. A top'l space X is regular if it is T_1 and

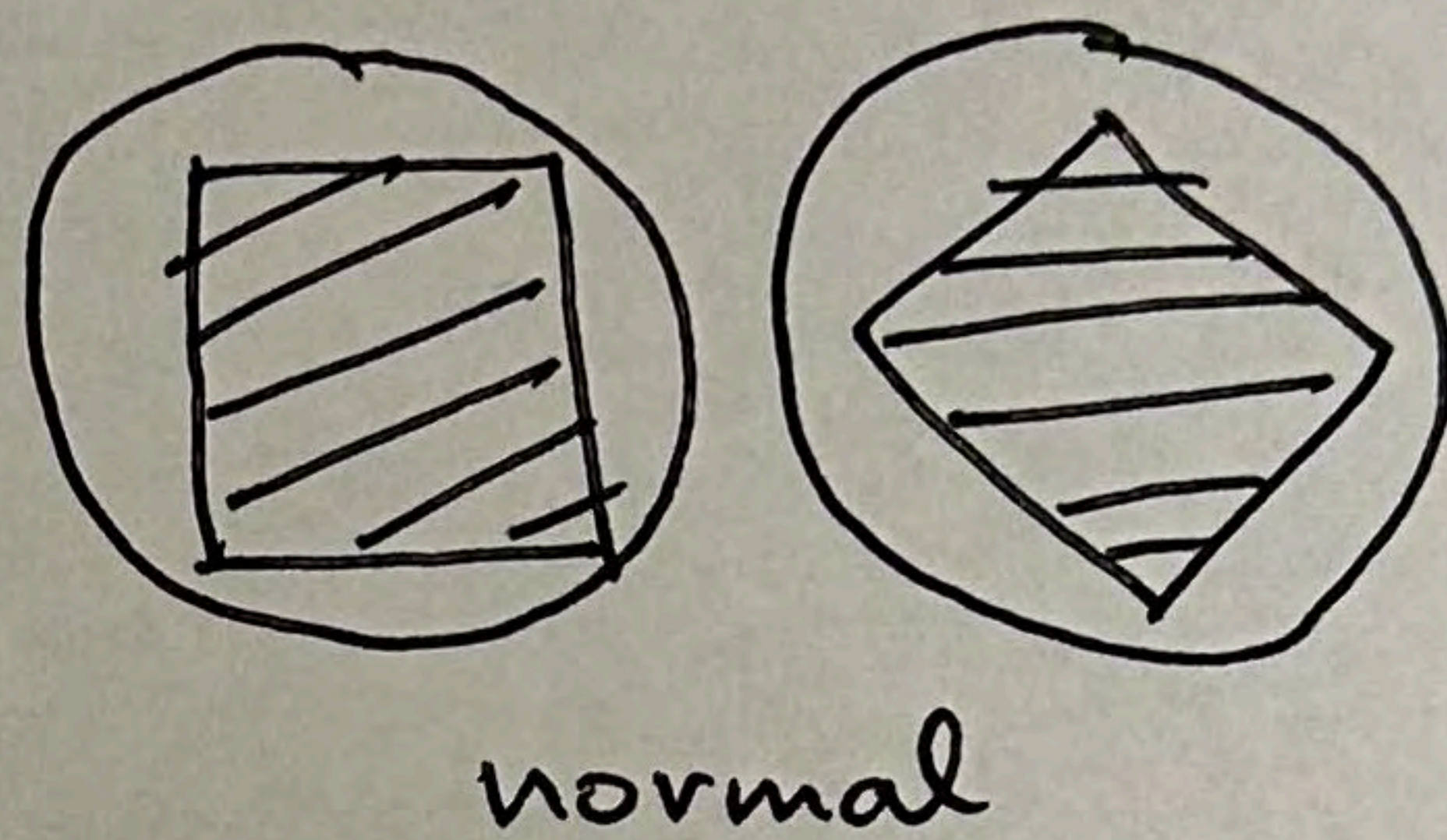
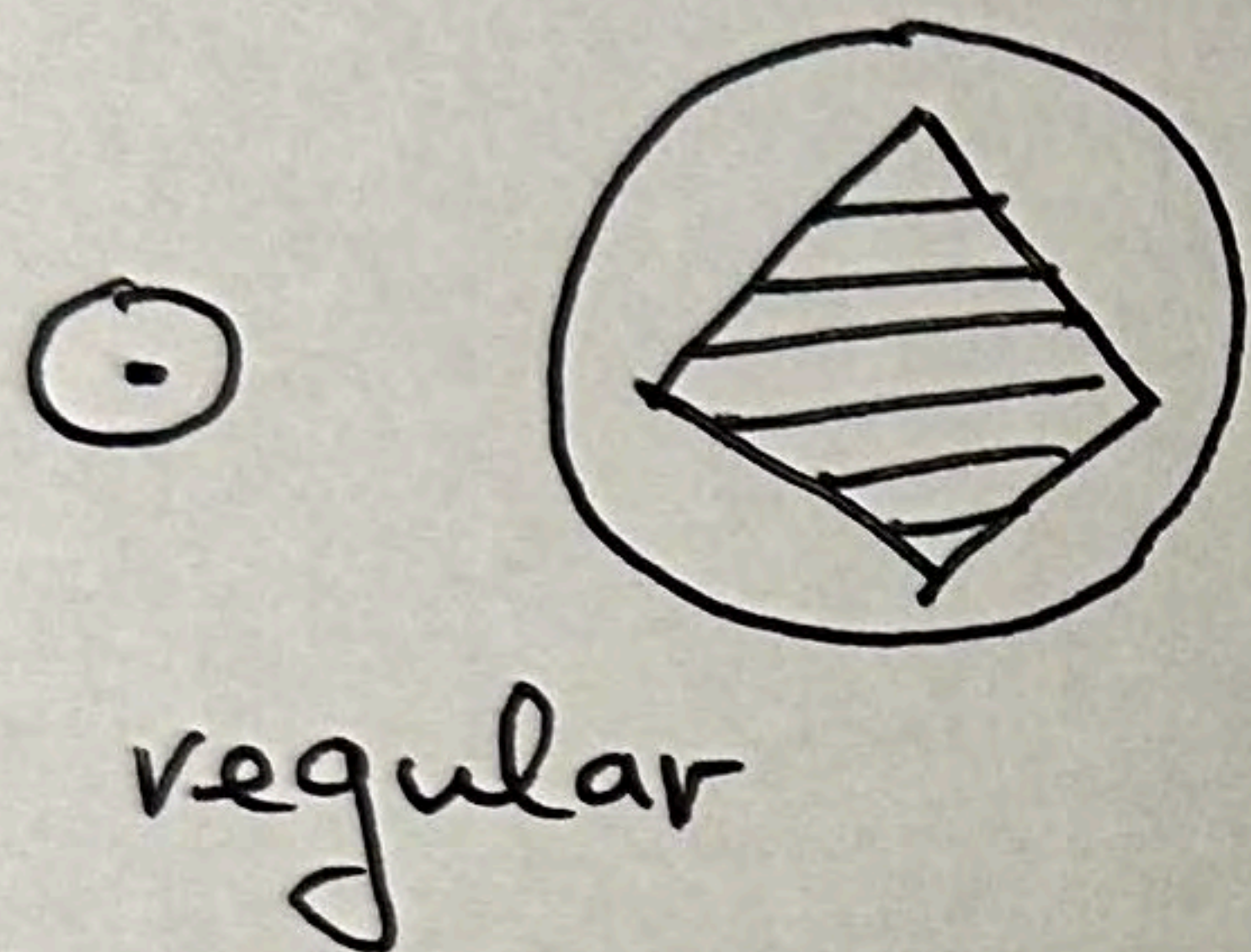
$$\forall x \in X \quad \forall F \subseteq X \text{ clsd } x \notin F \Rightarrow \exists \text{ open } U \ni x, V \supseteq F \quad U \cap V = \emptyset$$

$$(\Leftrightarrow) \forall x \in U \subseteq X \quad \exists \text{ open } V \text{ s.t. } x \in V \subseteq \bar{V} \subseteq U$$

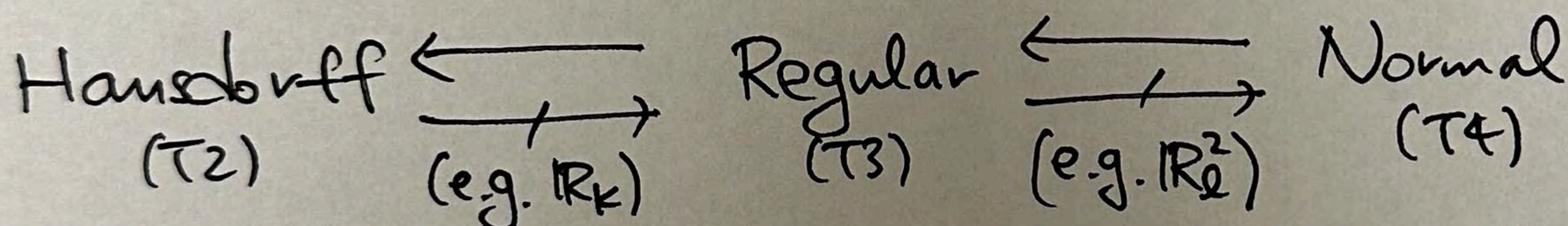
A top'l space X is normal if it is T_1 and

$$\forall F, G \subseteq X \text{ clsd } F \cap G = \emptyset \Rightarrow \exists \text{ open } U \supseteq F, V \supseteq G \quad U \cap V = \emptyset$$

$$(\Leftrightarrow) \forall F \subseteq U \subseteq X \quad \exists \text{ open } V \text{ s.t. } F \subseteq V \subseteq \bar{V} \subseteq U$$



Rmk.



Thm Subspaces/products of Hausdorff/Regular spaces is Hausdorff/Regular.

Rmk $\mathbb{R}_2$ is normal but $\mathbb{R}_2^2$ is not.

If X has the property that all its subspaces are normal, it is called completely normal (T_5).

$\bar{S}_1 \times \bar{S}_2$ is normal but not completely normal (Example 2 on p. 203)

§32. Normal Spaces.

Thm. A 2^{nd} countable regular space is normal.

pf) Let A, B be disjoint closed sets. By regularity, each $x \in A$ has an open nbhd O whose closure $\bar{O}$ doesn't intersect B . Let $\{U_n\}$ be a collection of basis elts whose closure doesn't intersect B . Then $\{U_n\}$ covers A . Likewise, let $\{V_n\}$ be a collection of basis elts whose closure doesn't intersect A . Then $\{V_n\}$ covers B .

Let $U'_n = U_n \setminus \bigcup_{i=1}^n \bar{V}_i$, $V'_n = V_n \setminus \bigcup_{i=1}^n \bar{U}_i$. Then $\bigcup_n U'_n$, $\bigcup_n V'_n$ are the desired nbhds.

Thm. A metrizable space is normal.

Thm. Every cpt Hdf space is normal.

Fact. Every order topology is normal.

§ 33. The Urysohn lemma.

Urysohn lemma.

Let X be a normal space, $A, B \subseteq X$ disjoint closed subsets.

Then $\exists$ conti map $f: X \rightarrow [0, 1]$ s.t. $f(A) \subseteq \{0\}$, $f(B) \subseteq \{1\}$.

P^e) Let $U_1 = X \setminus B$. Let $A \subseteq U_0 \subseteq \bar{U}_0 \subseteq U_1$, U_0 open. Repeat:

$$A \subseteq U_0 \subseteq \bar{U}_0 \subseteq U_{1/2} \subseteq \bar{U}_{1/2} \subseteq U_1 \subseteq X \setminus B.$$

Repeat:

$$A \subseteq U_0 \subseteq \bar{U}_0 \subseteq U_{1/4} \subseteq \bar{U}_{1/4} \subseteq U_{1/2} \subseteq \bar{U}_{1/2} \subseteq U_{3/4} \subseteq \bar{U}_{3/4} \subseteq U_1 \subseteq X \setminus B.$$

So for each number $r = \frac{m}{2^k}$, $r \in [0, 1]$, we get an open set U_r , so that $r < s \rightarrow \bar{U}_r \subseteq U_s$. Define

$$U_p = \emptyset, \quad p < 0,$$

$$U_p = X, \quad p > 1.$$

Define

$$f(x) = \inf \{p : x \in U_p\} \in [0, 1].$$

Then $f = 0$ on A , $f = 1$ on B , and

$$x \in \bar{U}_r \rightarrow f(x) \leq r,$$

$$x \notin U_r \rightarrow f(x) \geq r.$$

So if $f(x_0) \in (c, d)$, $c < p < f(x_0) < q < d$, then $U_q \setminus \bar{U}_p$ is a nbhd of x_0 , $f(U_q \setminus \bar{U}_p) \subseteq [p, q]$.

$c.f$) T_0 : distinct pts have distinct nbhds

T_1 : pts are closed.

T_2 : Hausdorff

T_3 : regular

$T_{3\frac{1}{2}}$: completely regular ($X \setminus B$ closed $\rightarrow \exists$ conti $f: X \rightarrow [0, 1]$ $f(x) = 0$, $f(p) = 1$)

T_4 : normal

T_5 : completely normal (all subspaces are normal,

$\Leftrightarrow A \cap \bar{B} = \bar{A} \cap B = \emptyset$ implies separation)

"Trennungssaxiom"

(It can be checked that subspcs and products of $T_{3\frac{1}{2}}$ are $T_{3\frac{1}{2}}$.)

§34. The Urysohn Metrization Theorem.

A regular space X w/ a countable basis is metrizable.

Proof) Let $\{U_n\}$ be a countable basis for $\mathcal{T}$.

For n, m s.t. $\overline{U_m} \subseteq U_n$, let $f_{n,m}: X \rightarrow [0,1]$ be a continuous function so that $f_{n,m} = 1$ on $\overline{U_m}$ and $f_{n,m} = 0$ on $X \setminus U_n$. (by Urysohn lemma).

Then the topology $\mathcal{T}$ on X is the coarsest topology on X for which all the functions $f_{n,m}$ are continuous. Indeed, denoting by $\mathcal{T}'$ the coarsest such topology, we have $\mathcal{T}' \subseteq \mathcal{T}$ b/c $f_{n,m}$ are continuous w.r.t. $\mathcal{T}$. On the other hand, if $x \in U_n$, choose an m s.t. $x \in \overline{U_m} \subseteq U_n$. Then $f_{n,m}^{-1}([0,1]) \in \mathcal{T}'$ and $x \in f_{n,m}^{-1}([0,1]) \subseteq U_n$. So $U_n \in \mathcal{T}'$ and $\mathcal{T}' = \mathcal{T}$.

By Problem 5(b) of Problem Set 1, the map $\Phi: X \rightarrow [0,1]^\omega$, $x \mapsto (f_{n,m}(x))$, is an embedding. Since $[0,1]^\omega$ is metrizable, so is X . $\square$

Rmk. $[0,1]^\omega$ has metric $d(\{x_k\}, \{y_k\}) = \sum_{k=1}^{\infty} 2^{-k} |x_k - y_k|$,

so the pull-back of this metric by Φ defines a metric for $(X, \mathcal{T})$:

$$d_X(x, y) = \sum_{\substack{n,m=1 \\ \overline{U_m} \subseteq U_n}}^{\infty} 2^{-n-m} |f_{n,m}(x) - f_{n,m}(y)|.$$

We could have verified this as a metric.

Similarly, we can prove the following embedding thm, which characterizes completely regular spaces.

Thm. A space is completely regular iff it embeds into some $[0,1]^J$.

(p1) $\Leftarrow$ products and subspaces of completely regular spaces are completely regular.

$\Rightarrow$ Let $J = \{(x, U) : x \in U \in \mathcal{T}\}$. For each $(x, U) \in J$, choose $f_{x,U}: X \rightarrow [0,1]$ s.t. $f_{x,U}(x) = 1$, $f_{x,U} = 0$ on $X \setminus U$. Take $\Phi: X \rightarrow [0,1]^J$, $y \mapsto (f_{x,U}(y))_{(x,U) \in J}$.

§35. The Tietze Extension Theorem.

Let X be a normal space and $A \subseteq X$ a closed subsp.

Any continmap $f: A \rightarrow [a, b]$ admits a conti extension $g: X \rightarrow [a, b]$
(or (a, b))

pf)

Lemma. Given conti $\psi: A \rightarrow [-r, r]$, there is conti $\phi: X \rightarrow [-\frac{r}{3}, \frac{r}{3}]$
s.t. $|\phi - \psi| \leq \frac{2}{3}r$ on A .

pf) Take $F_1 = \psi^{-1}[-r, -\frac{r}{3}]$, $F_2 = \psi^{-1}[\frac{r}{3}, r]$. These are disj. closed subsets of A , hence of X . By Urysohn lemma, there is conti $\phi: X \rightarrow [-\frac{r}{3}, \frac{r}{3}]$ w/ $\phi|_{F_1} = -\frac{r}{3}$, $\phi|_{F_2} = \frac{r}{3}$. $\square$

Applying the Lemma inductively, we have a sequence $\{h_n: X \rightarrow [-\frac{2^{n-1}}{3^n}, \frac{2^{n-1}}{3^n}]\}$
to $f: A \rightarrow [-1, 1]$

such that $|f - h_1 - \dots - h_n| \leq (\frac{2}{3})^n$ on A . Take $g = \sum_{n=1}^{\infty} h_n$. $\blacksquare$

If given $f: A \rightarrow (-1, 1)$, use above to extend to $g_0: X \rightarrow [-1, 1]$.

Then A , $g_0^{-1}(\{\pm 1\})$ are disj closed sets, so $\exists \eta: X \rightarrow [0, 1]$ conti s.t.

$$\eta = 1 \text{ on } A, \quad \eta = 0 \text{ on } g_0^{-1}(\{\pm 1\}).$$

Take $g = \eta g_0$. $\square$

§36. Imbeddings of Manifolds.

Def. Let X be a topol sp.

- The support of a function $\phi: X \rightarrow \mathbb{R}$ is $\overline{\{x \in X: \phi(x) \neq 0\}} = \text{spt } \phi$.
- A partition of unity dominated by a finite open cover $\{U_i\}_{i=1}^n$ of X is a family $\{\phi_i: X \rightarrow [0,1]\}_{i=1}^n$ of cont' fns s.t.
 - (1) $\text{spt } \phi_i \subseteq U_i$
 - (2) $\sum_{i=1}^n \phi_i = 1$ on X .

Thm (Existence of finite partitions of unity.)

Let $\{U_1, \dots, U_n\}$ be a finite open covg of the normal space X .

There is a partition of unity dominated by $\{U_1, \dots, U_n\}$.

pf) Since $X \setminus (U_2 \cup \dots \cup U_n) \subseteq U_1$, there is V_1 open s.t.

$$X \setminus U_2 \cup \dots \cup U_n \subseteq V_1 \subseteq \bar{V}_1 \subseteq U_1.$$

So $\{V_1, U_2, \dots, U_n\}$ is still a cover of X w/ $\bar{V}_1 \subseteq U_1$.

Likewise, we obtain V_2 open s.t. $\{V_1, V_2, U_3, \dots, U_n\}$ is a cover of X w/ $\bar{V}_2 \subseteq U_2$.

Repeating, we obtain a cover $\{V_1, \dots, V_n\}$ of X w/ $\bar{V}_i \subseteq U_i$.

Repeating, we obtain a cover $\{W_1, \dots, W_n\}$ of X w/ $\bar{W}_i \subseteq V_i$.

By Urysohn, there is a conti $\tilde{\phi}_i: X \rightarrow [0,1]$ w/ $\tilde{\phi}_i|_{\bar{W}_i} = 1$, $\tilde{\phi}_i|_{X \setminus V_i} = 0$.

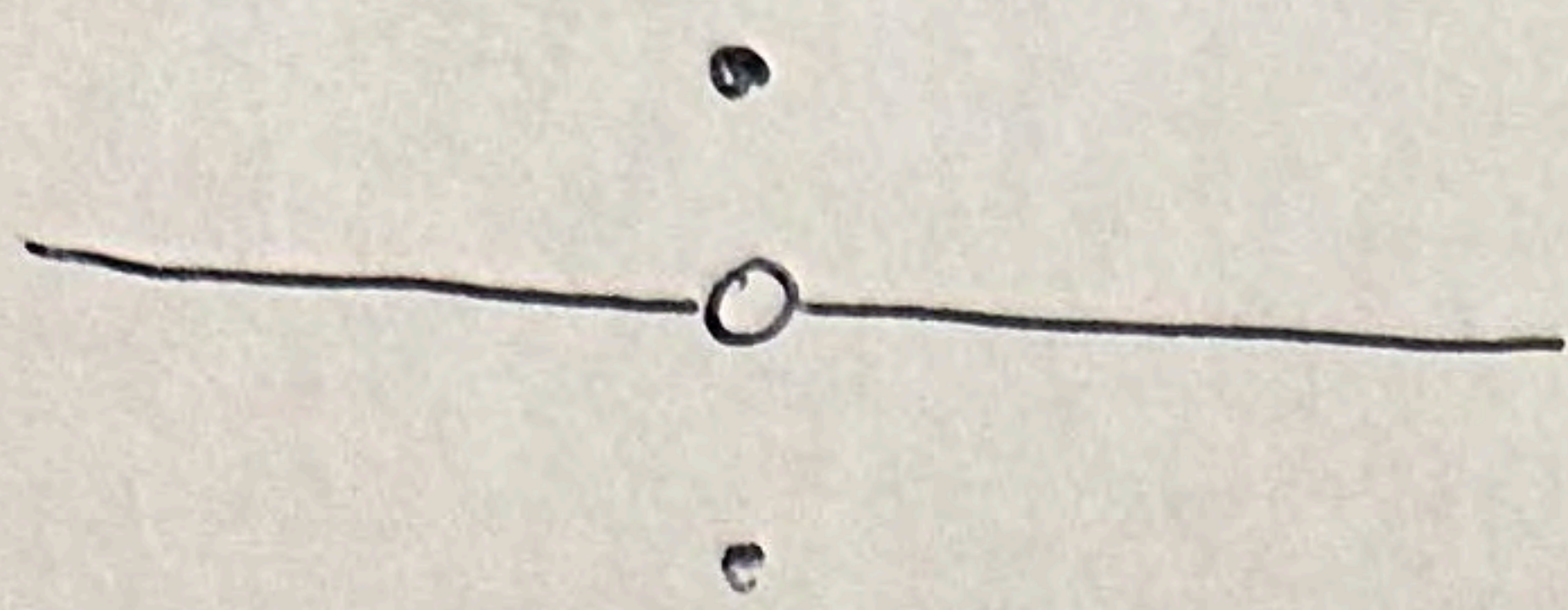
Then $\text{spt } \tilde{\phi}_i \subseteq \bar{V}_i$ and $\sum_{i=1}^n \tilde{\phi}_i > 0$.

Set $\phi_i = \tilde{\phi}_i / \sum_{j=1}^n \tilde{\phi}_j$. Then $\text{spt } \phi_i \subseteq \bar{V}_i \subseteq U_i$ and $\sum_{i=1}^n \phi_i = 1$. $\square$

Def. An n-manifold is a 2nd ctbl Hdf space locally homeo to $\mathbb{R}^n$.

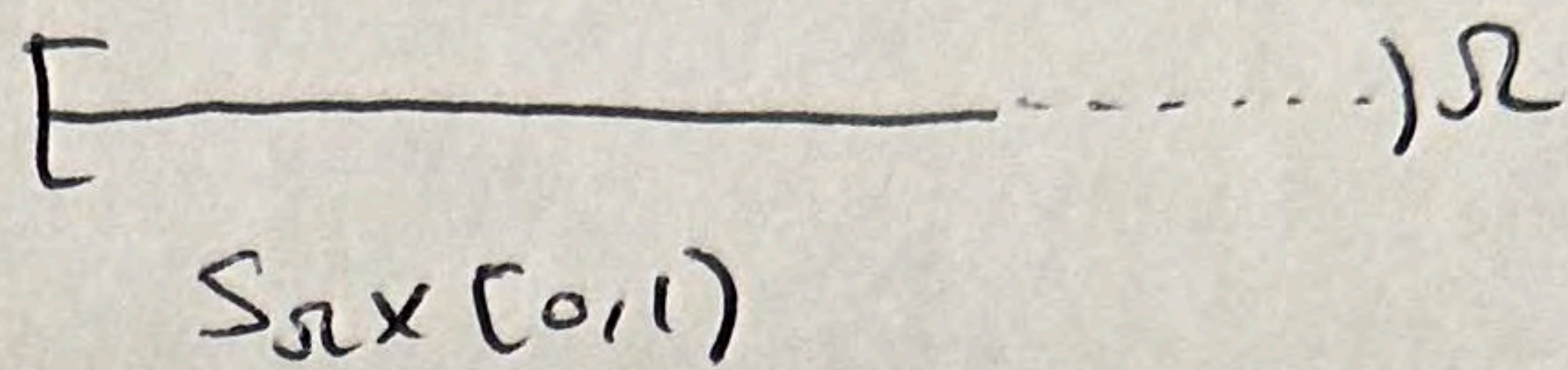
non-examples

line w/ double origin



not Hdf

long line

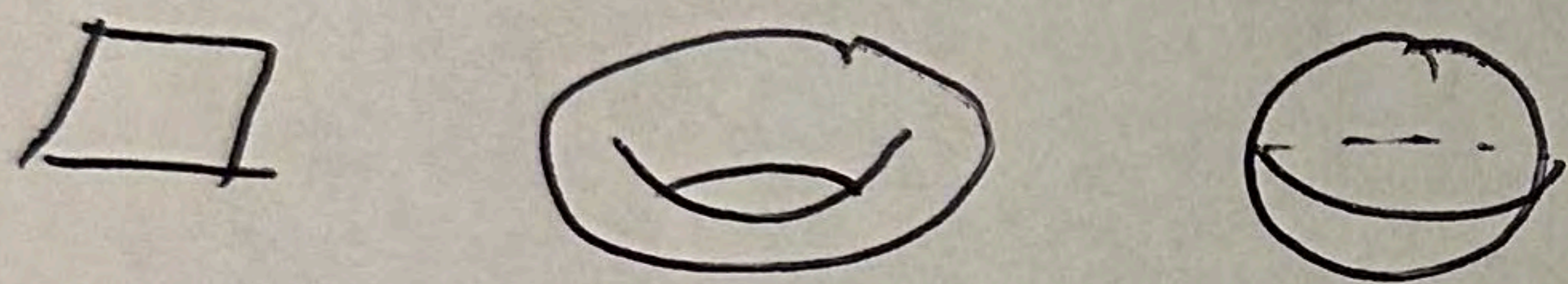


$S_{\mathbb{R}} \times [0,1)$

not 2nd ctbl.

examples

$\mathbb{R}^n, \mathbb{T}^n = (S^1)^n, S^n$



Thm If X is a compact n -mfd, it can be embedded in some $\mathbb{R}^N$.

pf) Find a finite cover $U_1, \dots, U_m$ of X w/ embeddings $h_i: U_i \rightarrow \mathbb{R}^n$.

Take subordinate partition of unity $\phi_1, \dots, \phi_m$. Define

$$\begin{aligned} \Phi: X &\longrightarrow \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{m \text{ times}} \times \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{m \text{ times}} \\ x &\longmapsto (\phi_1(x), \dots, \phi_m(x), \phi_1(x)h_1(x), \dots, \phi_m(x)h_m(x)) \end{aligned}$$

↑
interpreted to be 0
outside U_i .

This is an injective conti map of cpt X into Hdf $\mathbb{R}^N$, so it is an emb.

□

Rmk Bryant-Mio (1999) and Johnston (1999) have indep'tly shown you can take $N = 2n$.

In the smooth manifold category, you can make a smooth emb into $\mathbb{R}^{2n}$, this is the Whitney embedding thm (1936). We will learn to prove this into $\mathbb{R}^{2n+1}$ in Mal'09c, please come!

§37. The Tychonoff Theorem.

Tychonoff theorem.

An arbitrary product of compact spaces is compact in the product topology.
proof)

Step 1. (Reformulation of compactness)

X is compact

$\Leftrightarrow$ Any collection $\mathcal{F}$ of closed sets of X with the finite intersection property
(i.e., $\forall F_1, \dots, F_n \in \mathcal{F} \quad F_1 \cap \dots \cap F_n \neq \emptyset$)
has nonempty intersection

$\Leftrightarrow$ For any collection $\mathcal{A} = \{A_i\}_i$ of subsets of X w/ the F.I.P., $\bigcap_i \bar{A}_i \neq \emptyset$.

Step 2. (Upgrade to maximality using Zorn's lemma)

Lemma. X a set, $\mathcal{A} \subseteq \mathcal{P}(X)$ w/ F.I.P. Then $\exists \mathcal{D} \subseteq \mathcal{P}(X)$ s.t.

(1) $\mathcal{A} \subseteq \mathcal{D}$

(2) $\mathcal{D}$ has F.I.P.,

(3) Any collection $\mathcal{D}' \subseteq \mathcal{P}(X)$ satisfying (1) and (2) satisfies $\mathcal{D}' \subseteq \mathcal{D}$.

proof of Lemma.

Let

$\mathcal{A} = \{\mathcal{D}' \subseteq \mathcal{P}(X) : \mathcal{A} \subseteq \mathcal{D}' \text{ and } \mathcal{D}' \text{ has F.I.P.}\}$.

Partially order $\mathcal{A}$ by inclusion. If $\mathcal{B} \subseteq \mathcal{A}$ is a chain (i.e. totally ordered by inclusion), then $\mathcal{D}' = \bigcup \mathcal{B} \in \mathcal{A}$ is an upper bound for $\mathcal{B}$.

Here, $\mathcal{D}' \in \mathcal{A}$ b/c if $C_1, \dots, C_n \in \mathcal{D}'$, then $C_i \in \mathcal{D}_i' \in \mathcal{B}$, and if $\mathcal{D}_1' \subseteq \dots \subseteq \mathcal{D}_n'$, then $C_1, \dots, C_n \in \mathcal{D}_n'$ so that $C_1 \cap \dots \cap C_n \neq \emptyset$.

By Zorn's lemma, $\mathcal{A}$ has a maximal element. $\square$

Cor. (1) $C_1, \dots, C_n \in \mathcal{D} \rightarrow C_1 \cap \dots \cap C_n \in \mathcal{D}$.

(2) If $A \subseteq X$ s.t. $\forall C \in \mathcal{D} \quad A \cap C \neq \emptyset$, then $A \in \mathcal{D}$.

(3) If X top'l sp., then $x_0 \in \bigcap_{C \in \mathcal{D}} \bar{C} \Leftrightarrow \forall \text{ nbhd of } x_0 \text{ belongs to } \mathcal{D}$.

Cor. X compact

$\Leftrightarrow$ For any maximal collection $\mathcal{D}$ of subsets of X w/ the F.I.P.,
we have $\bigcap_{C \in \mathcal{D}} \bar{C} \neq \emptyset$.

Step 3. (Look coordinatewise)

Let $X = \prod_{\alpha \in I} X_\alpha$ be a product of compact spaces.

Let $\mathcal{D}$ be a maximal collection of subsets of X w/ the F.I.P.

Then $\forall \alpha \in I$, $\mathcal{D}_\alpha = \{\pi_\alpha(C) : C \in \mathcal{D}\} \subseteq \mathcal{P}(X_\alpha)$ w/ F.I.P.,

so $\exists x_\alpha \in \bigcap_{C \in \mathcal{D}_\alpha} \overline{\pi_\alpha(C)}$. We claim $\{x_\alpha\} \in \bigcap_{C \in \mathcal{D}} \overline{C}$. But

$$\{x_\alpha\} \in \bigcap_{C \in \mathcal{D}} \overline{C}$$

$\Leftarrow \forall$ basis nbhd of $\{x_\alpha\}$ belongs to $\mathcal{D}$

$\Leftarrow \forall$ subbasis nbhd $\pi_\beta^{-1}(U_\beta)$ of $\{x_\alpha\}$ intersects every elt of $\mathcal{D}$

$\Leftarrow \forall$ nbhd U_β of x_β in X_β intersects $\bigcap_{C \in \mathcal{D}} \overline{\pi_\beta(C)}$

$\Leftrightarrow x_\beta \in \bigcap_{C \in \mathcal{D}} \pi_\beta(C)$.

□

§38. The Stone-Čech Compactification

Defn. A compactification of a space X is a compact Hdf space Y containing X as a subspace s.t. $\bar{X} = Y$.

Two compactifications Y_1, Y_2 of X are equivalent if there is a homeo $h: Y_1 \rightarrow Y_2$ s.t. $h|_X = \text{id}_X$.

Rmk Lemma Let $A \subseteq X$, Z Hdf, $f: A \rightarrow Z$ conti.

Then $f: A \rightarrow Z$ has at most one conti extension $g: \bar{A} \rightarrow Z$.

How to make compactifications: via embeddings!

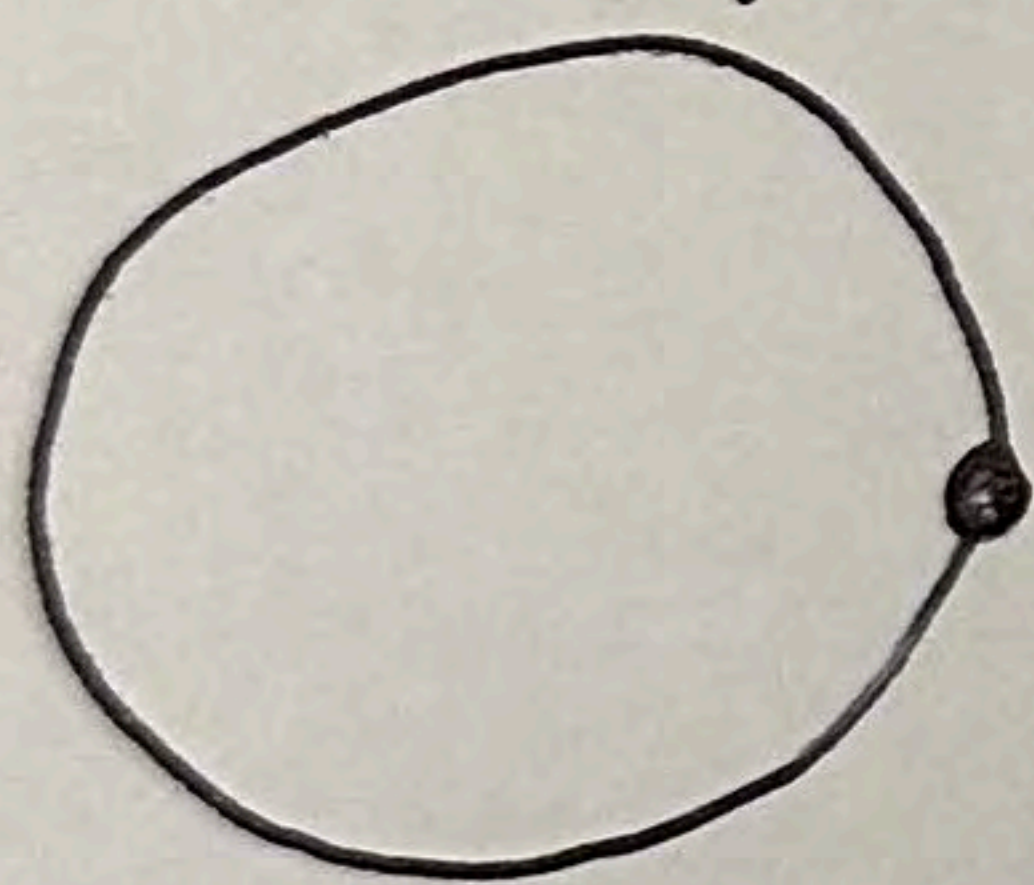
Lemma ^{38.3} X top'l sp, Z cpt Hdf, $h: X \rightarrow Z$ imbedding.

$\exists!$ compactification Y of X s.t. $\exists$ imbedding $H: Y \rightarrow Z$, $H|_X = h$.

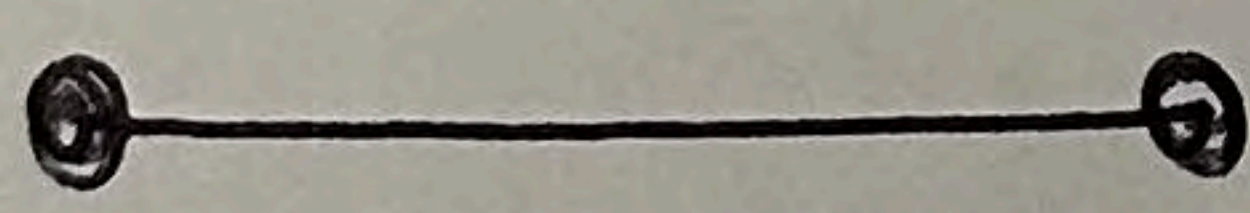
(Y is the compactification induced by the imb. h .)

pf) take Y to be a homeomorphic copy of $\overline{h(X)}$. $\square$

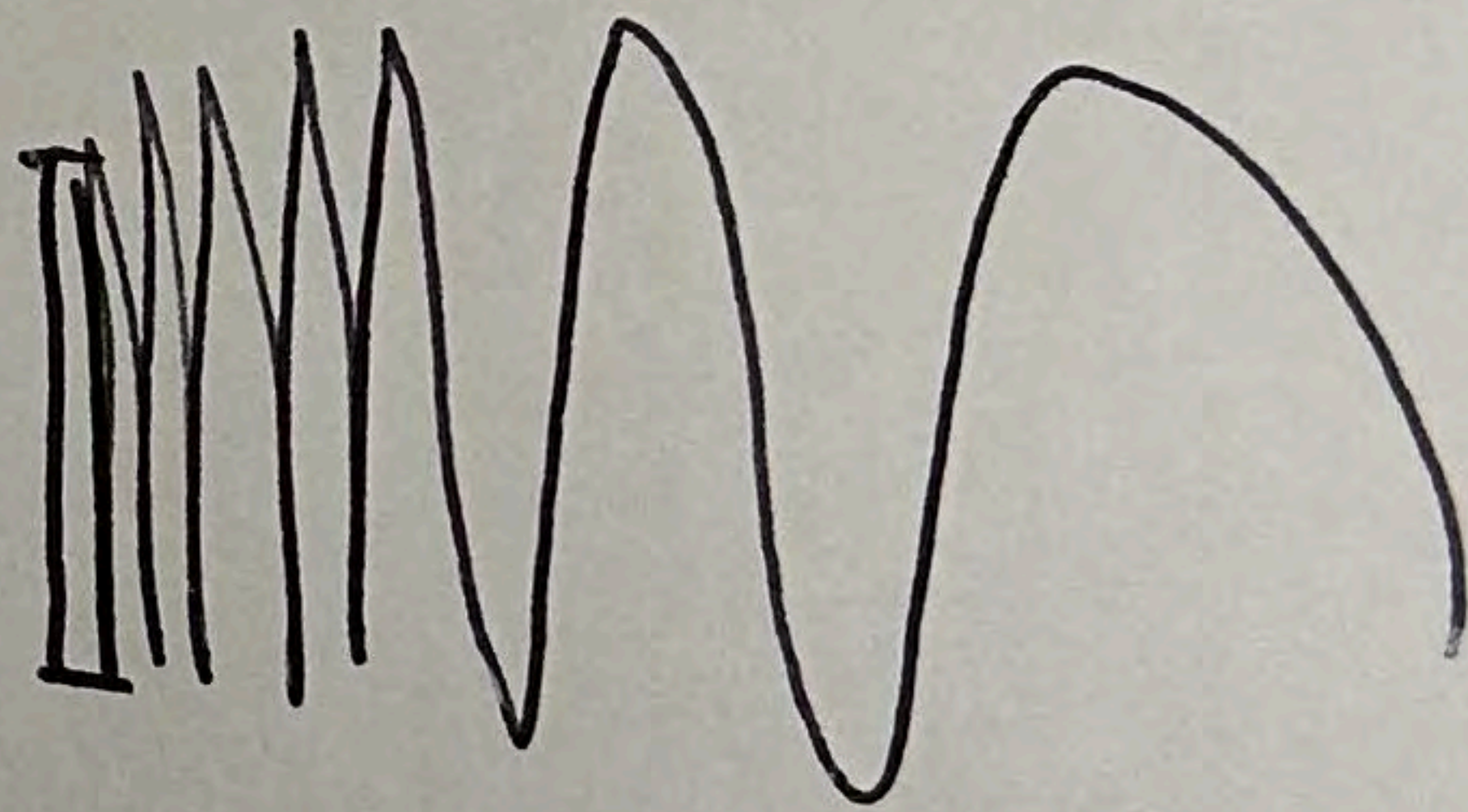
Examples of compactifications of $(0,1)$:



S^1



$[0,1]$



topologist's sine curve.

Thm 38.2 Let X be a completely regular space.

There is a compactification Y of X , called the Stone-Čech compactification $\beta(X)$ of X ,

s.t. any bdd conti map $f: X \rightarrow \mathbb{R}$ extends uniquely to a conti map $Y \rightarrow \mathbb{R}$.

pf) Let $\{f_\alpha: X \rightarrow [a_\alpha, b_\alpha] = I_\alpha\}_{\alpha \in J}$ be the collection of all bdd conti $\mathbb{R}$ -valued fns on X . Then

$$h: X \rightarrow \prod_{\alpha \in J} I_\alpha, \quad h = (f_\alpha)_{\alpha \in J}$$

is an embedding. Take Y as in Lem 38.3 s.t. $H: Y \rightarrow \prod_{\alpha \in J} I_\alpha$ emb.

Then $Y \xrightarrow{H} \prod_{\alpha \in J} I_\alpha \xrightarrow{\pi_\beta} I_\beta$ extends f_β . $\square$

Thm 38.4.

Let X be a completely regular space w/ Stone-Čech compactification $\beta(X)$.

If C Cpt Hdf, then any conti map $X \rightarrow C$ admits unique extension $\beta(X) \rightarrow C$.

pf) May embed C as a closed subsp of $[0,1]^J$. $\square$

Thm 38.5

Stone-Čech compactification is unique up to equivalence.

pf)
$$\begin{array}{ccc} X & \xrightarrow{i'} & \beta'(X) \\ \downarrow i & \searrow f & \\ \beta(X) & & \end{array} \quad \& \quad \begin{array}{ccc} X & \xrightarrow{i} & \beta(X) \\ \downarrow i' & \searrow g & \\ \beta'(X) & & \end{array}$$

$$\Rightarrow \begin{array}{ccc} X & \xrightarrow{i} & \beta(X) \\ \downarrow i & \searrow f & \\ \beta(X) & & \end{array} \quad \& \quad \begin{array}{ccc} X & \xrightarrow{i'} & \beta'(X) \\ \downarrow i' & \searrow g & \\ \beta'(X) & & \end{array}$$

 $f \circ g = \text{id}_{\beta(X)} \quad \& \quad g \circ f = \text{id}_{\beta'(X)}$

◦◦ The Stone-Čech compactification $\beta(X)$ of a completely regular space X is characterized by the fact that any conti map

$X \rightarrow C$, C cpt Hdf, uniquely extends to a conti map $\beta(X) \rightarrow C$.

Chapter 6. Metrization Theorems and Paracompactness

§39. Local Finiteness.

Urysohn metrization thm.

X regular, 2nd countable $\Rightarrow X$ metrizable.

The converse doesn't hold, as there are non-separable metric spaces.

Nagata-Smirnov Metrization Thm.

X regular, has a countably locally finite basis $\Leftrightarrow X$ metrizable.

The goal of the next two sections is to prove this.

Defn A collection $\mathcal{A}$ of subsets of a topol sp. X is locally finite if every pt in X has a nbhd that intersects only finitely many elts of $\mathcal{A}$.

$\mathcal{A}$ is countably locally finite if $\mathcal{A} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$, $\mathcal{A}_n$ loc. finite.

ex) $\mathcal{A} = \{(n, n+2) : n \in \mathbb{Z}\}$ loc. finite in $\mathbb{R}$.

$\mathcal{A} = \{(0, \frac{1}{n}) : n \in \mathbb{Z}_{>0}\}$ loc. finite in $(0,1)$, not in $[0,1]$.

Rmk. A subtle distinction between collection ^{$\mathcal{A}$} of subsets and an indexed family $\{A_\alpha\}_{\alpha \in J}$ of subsets of X is that the latter allows for repetition. So

$\mathcal{A} = \{\mathbb{R}\}$ is loc. finite in $\mathbb{R}$,

$\{\mathbb{R}\}_{\alpha \in \mathbb{Z}}$ is not loc. finite in $\mathbb{R}$.

$\{A_\alpha\}_{\alpha \in J}$ is locally finite
if $\forall$ pt has nbhd that intersects
 A_α for only finitely many α

The collection $\{A_\alpha : \alpha \in J\}$ is loc. finite iff $\{A_\alpha\}_{\alpha \in J}$ is loc. finite and for each $A \subseteq X$, $A = A_\alpha$ for only finitely many α .

Lemma 39.1 Let $\mathcal{A}$ be a locally finite collection of subsets of X . Then

(a) Any subcollection of $\mathcal{A}$ is locally finite.

(b) $\{\bar{A} : A \in \mathcal{A}\}$ is locally finite.

(c) $\overline{\bigcup_{A \in \mathcal{A}} A} = \bigcup_{A \in \mathcal{A}} \bar{A}$.

pf) (b) If U is an open set that intersects some $\bar{A}$, it must intersect A also. So if it intersects finitely many A 's, it should intersect finitely many $\bar{A}$'s.

(c) This is bc closure is a local operation and finite unions of closed sets are closed. More precisely, $\bigcup_{A \in \mathcal{A}} \bar{A} \subseteq \overline{\bigcup_{A \in \mathcal{A}} A}$ is obvious. Conversely, let $x \in \overline{\bigcup_{A \in \mathcal{A}} A}$.

Let U be an open nbhd of x that intersects only finitely many elts $A_1, \dots, A_n$ of $\mathcal{A}$. For any nbhd V of x , $U \cap V$ is a nbhd of x that intersects $\bigcup_{A \in \mathcal{A}} A$; by choice of U , it must intersect $\bigcup_{i=1}^n A_i$.

So any nbhd V of x intersects $\bigcup_{i=1}^n A_i$, i.e.,

$$x \in \overline{\bigcup_{i=1}^n A_i} = \bigcup_{i=1}^n \bar{A}_i \subseteq \bigcup_{A \in \mathcal{A}} \bar{A}.$$

Def. Given two collections $\mathcal{A}, \mathcal{B}$ of subsets of X , $\mathcal{B}$ is a refinement of $\mathcal{A}$ if $\forall B \in \mathcal{B} \exists A \in \mathcal{A} \text{ s.t. } B \subseteq A$.

$\mathcal{B}$ is an open/closed refinement if each elt of $\mathcal{B}$ is open/closed.

Lemma 39.2.

Any open covering $\mathcal{A}$ of a metrizable space X has a countably locally finite open refinement.

(Rmk. By Lemma 41.3, we can in fact take it to be locally finite.)

pf) Take a well-ordering $<$ on A . Define

$$S_n(u) := \{x: B(x, \frac{1}{n}) \subseteq u\} = \{x: d(x, X \setminus u) \geq \frac{1}{n}\}.$$

Then $\{S_n(u): u \in A, n \in \mathbb{N}\}$ covers X . Now take

$$T_n(u) := S_n(u) - \bigcup_{v < u} v.$$

Then

① $\{T_n(u): u \in A, n \in \mathbb{N}\}$ covers X . $\left(\because < \text{ is a well-ordering, so for given } x \in X, \text{ choose minimal } u \in A \text{ s.t. } x \in u. \right.$
Then $x \in T_n(u)$

② For fixed n , $u < v$, we have $d(T_n(u), T_n(v)) \geq \frac{1}{n}$,
so $\{T_n(u): u \in A\}$ is loc. finite.

But each $T_n(u)$ isn't open, so we take a small nbhd

$$E_n(u) := \text{Nbhd}_{1/3n}(T_n(u)) = \{x \in X: d(x, T_n(u)) < \frac{1}{3n}\}.$$

Then

① $\{E_n(u): u \in A, n \in \mathbb{N}\}$ is an open cover of X ,

② For $n \in \mathbb{N}$, $u < v$, $d(E_n(u), E_n(v)) \geq \frac{1}{3n}$, so $\{E_n(u): u \in A\}$ loc. fin.

□

Actually, we have created a countably locally discrete open refinement,
where A is locally discrete if $\forall x \in X \exists \text{ nbhd } U \ni x \text{ s.t. } U \text{ intersects}$
at most one elt of A .

§ 40. The Nagata-Smirnov Metrization Thm.

Defn A G_δ set in X is a countable intersection of open sets.

Gebiet Durchschnitt
"open set" "intersection"

An F_σ set in X is a countable union of closed sets

Fermé somme.
"closed" "union"

• examples of G_δ sets.

- an open set.
- singletons in a 1st ctbl T_1 space.
- closed sets in a metrizable space.

• $\mathbb{R}$ in S_2 is not G_δ .

• complements of G_δ set is F_σ , and vice versa.

Lemma 40.1

Let X be a regular space w/ a countably locally finite basis. Then

(a) every closed set is G_δ ($\Leftrightarrow$ every open set is F_σ)

(b) X is normal.

pt) (a) Let $\mathcal{U}$ be a basis, where $\mathcal{B}_n$ is locally finite. Let U be open, let

$$U_n = \bigcup \{B : B \in \mathcal{B}_n \text{ and } B \subseteq U\}.$$

By local finiteness,

$$\overline{U_n} = \bigcup \{\overline{B} : B \in \mathcal{B}_n \text{ and } B \subseteq U\} \subseteq U$$

while by regularity,

$$\bigcup U_n = U. \Rightarrow \bigcup \overline{U_n} = U.$$

(b) Let C, D be disjoint closed sets in X . By construction of (a),

$\exists$ open sets $U_n, n=1, \dots, \infty$, s.t. $\bigcup U_n = \bigcup \overline{U_n} = X \setminus D$,

$\exists$ open sets $V_n, n=1, \dots, \infty$, s.t. $\bigcup V_n = \bigcup \overline{V_n} = X \setminus C$.

Similar to the proof of [reg & 2nd ctbl $\rightarrow$ nrm], take

$$U'_n = U_n \setminus \bigcup_{i=1}^n \overline{V_i}, \quad V'_n = V_n \setminus \bigcup_{i=1}^n \overline{U_i}$$

$\Rightarrow \bigcup U'_n, \bigcup V'_n$ form separation of C and D .

Lemma 40.2

X normal, A closed $G_\delta \Rightarrow \exists$ conti $f: X \rightarrow [0,1]$ s.t. $f=0$ on A , $f>0$ on $X \setminus A$.
pf) Take $A = \bigcap_{n=1}^{\infty} U_n$. Take (by Urysohn) $f_n: X \rightarrow [0,1]$ s.t. $f_n=0$ on A , $f_n=1$ on $X \setminus U_n$.
Take $f = \sum_{n=1}^{\infty} 2^{-n} f_n$. □

Thm 40.3 (Nagata-Smirnov metrization thm)

X metrizable $\Leftrightarrow X$ reg & $\exists$ countably local-finite basis.

$\Rightarrow$] The open cng

$$A_m = \{B(x, \frac{1}{m}) : x \in X\}$$

has (by Lemma 39.2) a countably locally finite open refinement B_m .
($\neq$ cng)

Then $\bigcup_m B_m$ is a countably locally finite basis.

$\Leftarrow$] Let $\bigcup_n B_n$ be a basis of X , each B_n loc. finite. For $B \in B_n$,

$\exists$ conti $f_{n,B}: X \rightarrow [0, \frac{1}{n}]$ s.t. $f_{n,B} > 0$ on B , $f_{n,B} = 0$ on $X \setminus B$.

Let

$$J = \{(n, B) : n \in \mathbb{Z}_+, B \in B_n\} \subseteq \mathbb{Z}_+ \times B,$$

define $F: X \rightarrow [0,1]^J$ by

$$F(x) = (f_{n,B}(x))_{(n,B) \in J}.$$

Then, w/ $[0,1]^J$ given product topology, F is an embedding (cf §34).

Claim With $[0,1]^J$ given uniform metric topology, F is an embedding

pf) Since uniform topology is finer than product topology,

enough to show $F: X \rightarrow ([0,1]^J, \rho)$ conti. Fix $x_0 \in X$ and $\varepsilon > 0$.

For each $n \in \mathbb{N}$, only finitely many $f_{n,B}$'s are nonzero near x_0 ,
so $\exists$ nbhd V_n of x_0 s.t. $x \in V_n \Rightarrow |f_{n,B}(x) - f_{n,B}(x_0)| < \varepsilon$ for $B \in B_n$.

Take N s.t. $\frac{1}{N} < \varepsilon$. Then for $x \in V_1 \cap \dots \cap V_N$,

$$n \leq N, B \in B_n \Rightarrow |f_{n,B}(x) - f_{n,B}(x_0)| < \varepsilon$$

$$n > N, B \in B_n \Rightarrow |f_{n,B}(x) - f_{n,B}(x_0)| \leq \frac{1}{n} < \varepsilon.$$

so that $\rho(F(x), F(x_0)) \leq \varepsilon$. (end of proof of claim).

Since $([0,1]^J, \rho)$ metrizable, X metrizable.

□

§41. Paracompactness.

Defn. X is paracompact if every open cvg of X has a locally finite open refinement cvg.

Ex. $\mathbb{R}^n$ is paracompact.

$\therefore$ Let $\bar{B}_n$ = closed ball of radius n at the origin.

For each n , take a finite subcover of $\bar{B}_n$, intersect w/ $\bar{B}_{n-1}^c$,

call the resulting collection $\mathcal{C}_n$. Then $\bigcup \mathcal{C}_n$ is a loc. fin. open refinement cvg.

Thm 41.2 A closed subsp of a paracompact space is paracpt.

pf) Usual trick of throwing complement of the closed space into the cvg. $\square$

Thm 41.1 A paracpt Hdf space is normal.

pf) Let $a \notin B$, B clsd in X . By Hdf, $\forall b \in B \exists$ nbhd U_b s.t. $a \notin \bar{U}_b$.

Then $\{U_b : b \in B\} \cup \{X \setminus B\}$ is an open cvr of X , and by paracptness

$\exists$ locally finite subcover $\mathcal{C}$ of X that is an open refinement.

Take $\mathcal{D} = \{C \in \mathcal{C} : C \cap B \neq \emptyset\} \Rightarrow \mathcal{D}$ covers B , $\forall D \in \mathcal{D} \ a \notin \bar{D}$.

Then $V = \bigcup_{D \in \mathcal{D}} D$ is a nbhd of B s.t. $\bar{V} = \bigcup_{D \in \mathcal{D}} \bar{D}$ doesn't contain a .

This proves regularity. Normality is proven by replacing $\{a\}$ w/ A above. $\square$

non-examples.

A paracpt subsp of a Hdf space need not be clsd, e.g. $(0,1) \subseteq \mathbb{R}$.

A subsp of a paracpt space need not be paracpt, e.g. $S_2 \times S_2 \subseteq \bar{S}_2 \times \bar{S}_2$.

Lemma 41.3 (Ernest Michael 1953?)

Let X be regular. TFAE.

Every open cvg of X has a refinement that is:

(1) An open cvg of X and countably loc. finite.

(2) A cvg of X and locally finite.

(3) A clsd cvg of X and loc. finite.

(4) An open cvg of X and loc. finite.

pf) (4) $\Rightarrow$ (1)

(2) $\Leftarrow$ (1) $\Leftarrow$ (4)
(2) $\Rightarrow$ (3) $\Rightarrow$ (4)

(2) $\Rightarrow$ (3) is short:

if A an open cng of X , $B = \{U \text{ open} \mid \bar{U} \subseteq A \text{ for some } A \in \mathcal{A}\}$ covers X by regularity. Take locally finite refinement $\mathcal{C}$ of B that covers X . Then

$$\mathcal{D} = \{Z \mid C \in \mathcal{C}\}$$

is a locally finite (Lemma 39.1), covers X , refines A .

(1) $\Rightarrow$ (2)

Let A be an open cng of X w/ an open refinement $\bigcup_{n=1}^{\infty} B_n$ s.t. each B_n locally finite. Let

$$V_i = \bigcup_{U \in B_i} U,$$

for $n \in \mathbb{Z}_+$, $U \in B_n$ define

$$S_n(U) = U \setminus \bigcup_{i < n} V_i.$$

Then $\mathcal{C}_n = \{S_n(U) : U \in B_n\}$ refines B_n .

Claim $\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n$ is a locally finite refinement of A that covers X .

pf) Let $x \in X$. Take minimal N s.t. $x \in V_N$. Then for $U \in B_N$ s.t. $x \in U$, $x \in S_N(U)$. So $\mathcal{C}$ covers X . Also, for $i < N$, $\exists$ nbhd W_i of x that intersects B_i finitely. Then $W_1 \cap \dots \cap W_{N-1} \cap U$ intersects $\mathcal{C}$ finitely. $\square$

(3) $\Rightarrow$ (4).

Let A be an open cng of X . Take a loc. finite refinement B that covers X .

Consider the open cover of X s.t. each elt finitely intersects B .

Take locally finite refinement $\mathcal{C}$ of B that is a closed cover of X .

For $B \in B$, expand by taking

$$E(B) = X \setminus \underbrace{\bigcup_{\substack{C \in \mathcal{C} \\ C \cap B = \emptyset \\ \text{closed}}} C}_{\text{open}} \supseteq B.$$

Then $\{E(B) : B \in B\}$ is locally finite ($\because \forall x \in X$ has nbhd that intersects finitely many elts of $\mathcal{C}$, so since $\mathcal{C}$ is a cover of X , enough to show each $C \in \mathcal{C}$ intersects finitely many $E(B)$, $B \in B$. But if C intersects $E(B)$, it intersects B).

For $B \in B$, let $F(B) \in \mathcal{A}$ s.t. $B \subseteq F(B)$. Then

$\{E(B) \cap F(B) : B \in B\}$ is a loc. finite open refinement of A that covers X . $\square$

Thm 41.4 Every metrizable space is paracompact.

pf) Lemmas 39.2 and 41.3 $\square$

Thm 41.5 Every regular Lindelöf space is paracpt.

pf) A fortiori, (i) of Lemma 41.3 holds.

Ex. $\mathbb{R}_\ell$ regular & Lindelöf so it is paracpt.

But $\mathbb{R}_\ell \times \mathbb{R}_\ell$ Hdf not normal, so not paracpt.

$\therefore$ product of paracpt need not be paracpt.

Ex $\mathbb{R}^\omega$ is paracpt in product & uniform topologies ($\because$ metrizable)
(also in box top under continuum hypothesis.)

Ex $\mathbb{R}^J$ Hdf not normal. $\Rightarrow$ not paracpt.

Lemma 41.6 (precise refinement)

X paracpt Hdf. w/ open cover $\{U_\alpha\}_{\alpha \in J}$. $\exists$ loc. finite open cov $\{V_\alpha\}_{\alpha \in J}$ s.t. $\bar{V}_\alpha \subseteq U_\alpha$.

pf) $\mathcal{A} = \{A \text{ open} \mid \bar{A} \subseteq U_\alpha \text{ for some } \alpha \in J\}$ covers X by regularity.

By paracptness, $\exists$ loc. finite open refinement $\{B_\beta\}_{\beta \in K}$ of $\mathcal{A}$ that covers X .

Take $f: K \rightarrow J$ s.t. $\forall \beta \in K \quad \bar{B}_\beta \subseteq U_{f(\beta)}$. Define

$$V_\alpha = \bigcup \{B_\beta \mid f(\beta) = \alpha\}.$$

Then $\bar{V}_\alpha \subseteq U_\alpha$, $\{V_\alpha\}_{\alpha \in J}$ loc. finite, covers X . $\square$.

Defn. (Partition of unity).

Let $\{U_\alpha\}_{\alpha \in J}$ be an open covg of X . A partition of unity on X

dominated by $\{U_\alpha\}_{\alpha \in J}$ is an indexed family of cont' fns $\phi_\alpha: X \rightarrow [0,1]$ s.t.

(1) $\text{spt } \phi_\alpha \subseteq U_\alpha$

(2) $\{\text{spt } \phi_\alpha\}$ loc fin.

(3) $\sum_\alpha \phi_\alpha = 1$. (this makes sense by (2)).

Thm 41.7 X paracpt Hdf space. $\{U_\alpha\}_{\alpha \in J}$ open cving of X .

Then $\exists$ p.o.u. on X den by $\{U_\alpha\}_{\alpha \in J}$.

pf) By Lemma 41.6, $\exists$ loc. finite open covers $\{W_\alpha\}_{\alpha \in J}$, $\{V_\alpha\}_{\alpha \in J}$ of X
s.t. $\overline{W_\alpha} \subseteq V_\alpha \subseteq \overline{V_\alpha} \subseteq U_\alpha$. Since X normal, $\exists$ conti $\psi_\alpha: X \rightarrow [0,1]$
s.t. $\psi_\alpha = 1$ on $\overline{W_\alpha}$, $\psi_\alpha = 0$ on $X \setminus V_\alpha$. Then

$$\Psi = \sum_{\alpha \in J} \psi_\alpha(x)$$

is a locally finite sum and so is continuous. Take

$$\phi_\alpha = \psi_\alpha / \Psi. \quad \square$$

Thm 41.8 X paracpt Hdf space, $\mathcal{C} \subseteq \mathcal{P}(X)$. For $\forall C \in \mathcal{C}$, $\varepsilon_C > 0$ given.

If $\mathcal{C}$ loc. fin, $\exists f: X \rightarrow (0, \infty)$ conti s.t. $f \leq \varepsilon_C$ on C .

pf) Take $\{U_\alpha\}_{\alpha \in J}$ open cover of X s.t. each U_α finitely intersects $\mathcal{C}$.

Take subordinate p.o.u. $\{\phi_\alpha\}$. For $\alpha \in J$, set

$$\delta_\alpha = \min \{ \varepsilon_C : U_\alpha \text{ intersects } C \in \mathcal{C} \}.$$

$$\text{Take } f = \sum \delta_\alpha \phi_\alpha.$$

§42. The Smirnov Metrization Thm.

Defn X is locally metrizable if every pt $x \in X$ has a nbhd U that is metrizable.

Thm 42-1 (Smirnov Metrization Thm).

X metrizable $\Leftrightarrow X$ paracpt, Hdtf, locally metrizable.

pf) $\Leftarrow$ By Nagata-Smirnov, enough to show $\exists$ countably locally finite basis.

Cover X by metrizable open sets, take a locally finite open refinement $\mathcal{C}$ that covers X . For $C \in \mathcal{C}$, let d_C be a metric on C .

Take the covering of X :

$$A_m = \{ B_C(x, \frac{1}{m}) \mid x \in C \in \mathcal{C} \}, \quad m \in \mathbb{Z}_+.$$

Let D_m be a locally finite open refinement of A_m that covers X .

Then $\bigcup_m D_m$ is countably locally finite and is a basis.

$\because x \in \bigcup_m D_m \subseteq X$, there are finitely many $C_1, \dots, C_n \in \mathcal{C}$ s.t. $x \in C_i$.

Then $\exists \varepsilon_i > 0$ $B_{C_i}(x, \varepsilon_i) \subset \bigcup_n C_n$, take m s.t. $\frac{2}{m} < \min\{\varepsilon_1, \dots, \varepsilon_n\}$.

Since D_m covers X , $\exists D \in D_m$ s.t. $x \in D$, and $D \subseteq B_C(y, 1/m)$ for some $C \in \mathcal{C}$, $y \in C$. Then $x \in C$, so $C = C_i$ for some i . Then

$$x \in D \subset B_C(y, 1/m) \subseteq B_{C_i}(x, \varepsilon_i) \subseteq U_i.$$

□

Chapter 7. Complete Metric Spaces and Function Spaces.

§43. Complete Metric Spaces.

0. Preliminaries

Def A metric space (X, d) is complete if every Cauchy seq in X converges.

Rmk A closed subspace of a complete metric space is complete.

- A complete subspace of a metric space is closed.

- If $\bar{d} = \min\{d, 1\}$, then (X, d) complete iff $(X, \bar{d})$ complete.

Ex. $\mathbb{R}^n$ is complete in its l_p^n metric.

- $(\mathbb{R}^\omega, \text{prod})$ is complete in the metric $d(\{x_i\}_{i=1}^\infty, \{y_i\}_{i=1}^\infty) = \sum 2^{-n} \bar{d}(x_i, y_i)$.

1. Uniform metric.

Def. If (Y, d) is a metric space and J a set, the uniform metric on Y^J is

$$\bar{p}(f, g) = \sup\{\bar{d}(f(\alpha), g(\alpha)) : \alpha \in J\}.$$

Thm 43.5 If (Y, d) complete, then $(Y^J, \bar{p})$ is complete.

pf) If $\{f_n\}_{n=1}^\infty \subseteq Y^J$ Cauchy, then $\{f_n(\alpha)\}_{n=1}^\infty \subseteq Y$ uniformly Cauchy, $\rightarrow f(\alpha)$. $\square$

Def. X topol space, (Y, d) metric space.

$$\mathcal{C}(X, Y) = \{f \in Y^X : f \text{ conti}\}.$$

$$\mathcal{B}(X, Y) = \{f \in Y^X : f \text{ bdd}\} \quad (\text{i.e. } \text{diam } f(X) < \infty).$$

Thm $\mathcal{C}(X, Y), \mathcal{B}(X, Y)$ closed in Y^X .

complete if Y complete.

pf) unif limits of conti fns are conti, unif limits of bdd fns are bdd. $\square$

Def Define the sup metric on $\mathcal{B}(X, Y)$ by

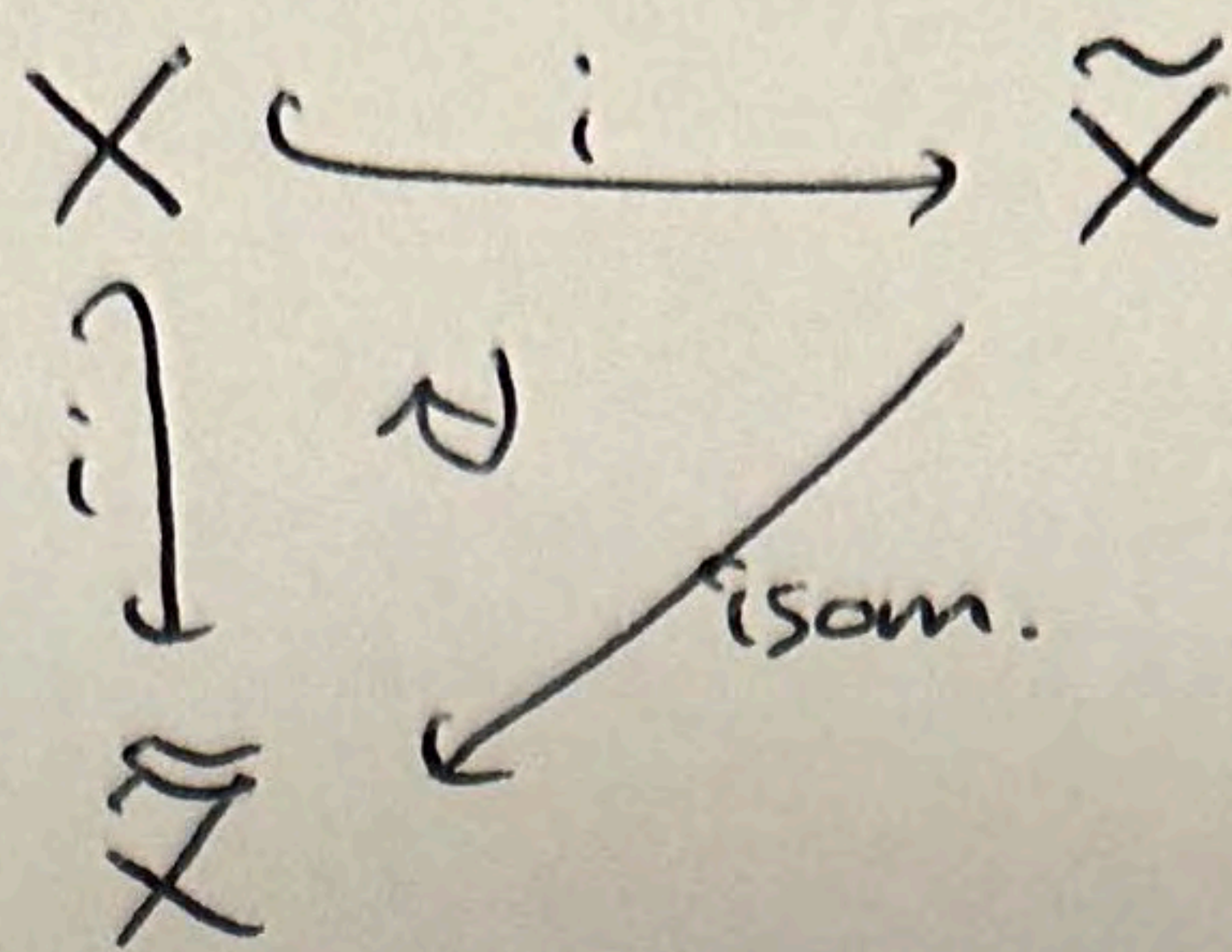
$$p(f, g) = \sup\{d(f(x), g(x)) : x \in X\}, \quad f, g \in \mathcal{B}(X, Y).$$

Rmk. $\bar{p} = \min\{p, 1\}$.

- If X cpe, $\mathcal{C}(X, Y) \subseteq \mathcal{B}(X, Y)$, so p def'd on $\mathcal{C}(X, Y)$.

2. Completions.

- If X is a metric space, there is a metric space $\tilde{X} \supseteq X$ isometrically containing X s.t. $\tilde{X} = \bar{X}$. Any two such completions are isometric:



- Completions exist: let $\tilde{X} = \{\text{Cauchy seqs in } X\} / \sim$, where two Cauchy seqs $\{x_n\}_n, \{y_n\}$ are equiv. if $d(x_n, y_n) \rightarrow 0$ as $n \rightarrow \infty$. Define distance between two (equiv. classes of) Cauchy seqs $\{x_n\}, \{y_n\}$ as

$$d(\{x_n\}, \{y_n\}) = \lim_{n \rightarrow \infty} d(x_n, y_n).$$

- One can also induce completions using isometric embeddings.

If $h: X \rightarrow Y$ is an isometric embedding, then $\overline{h(X)}$ is a completion of X .

- One may also prove existence of completion via the following universal embedding thm.

Thm 43.7 A metric space (X, d) isometrically embeds into the complete metric space $(B(X, \mathbb{R}), \rho)$.
(Fréchet 1910)
 (Kuratowski 1935)

pf) $X \longrightarrow B(X, \mathbb{R}) \quad x_0 \in X \text{ fixed.}$

$$a \mapsto (x \mapsto d(x, a) - d(x, x_0)),$$

□

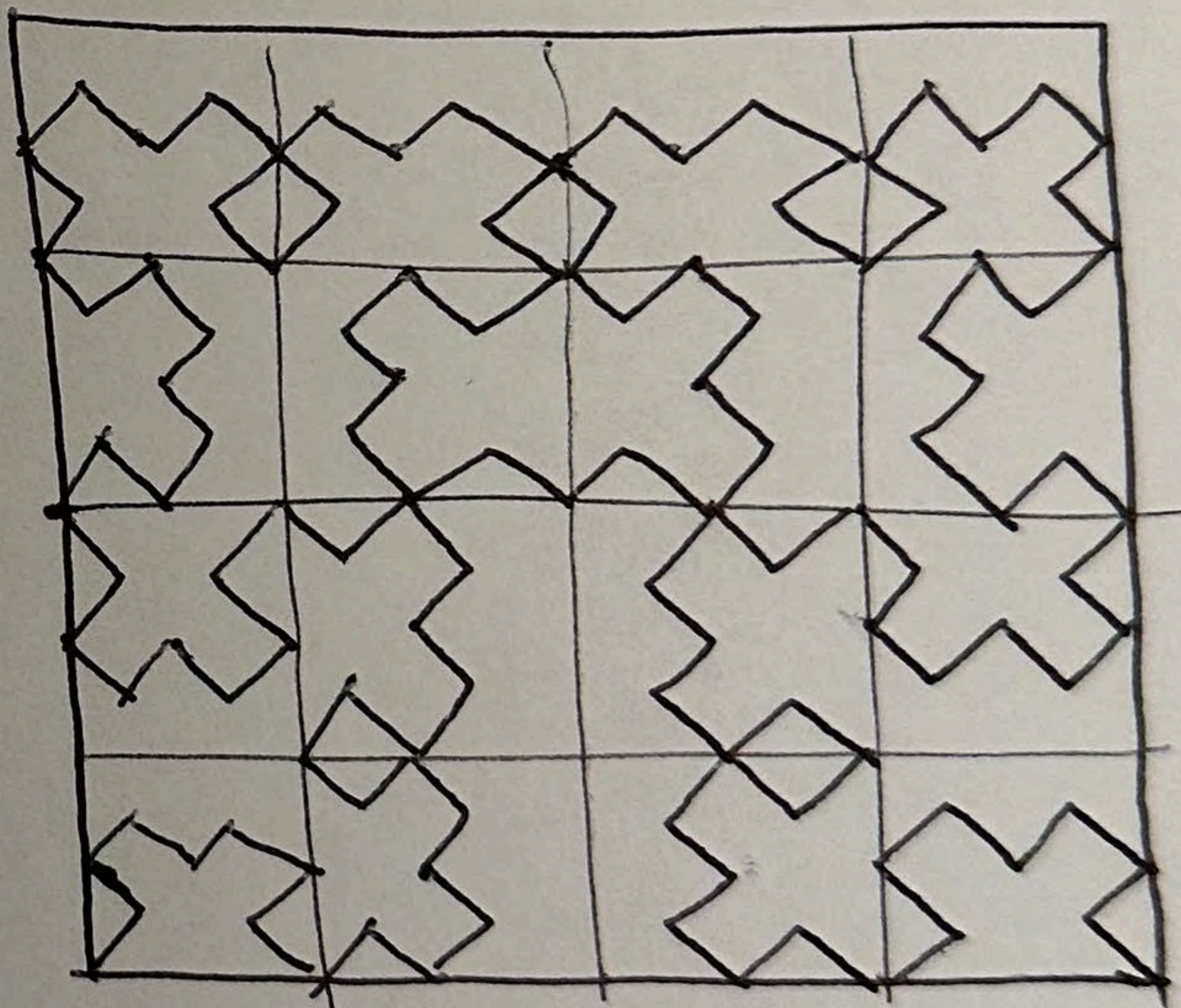
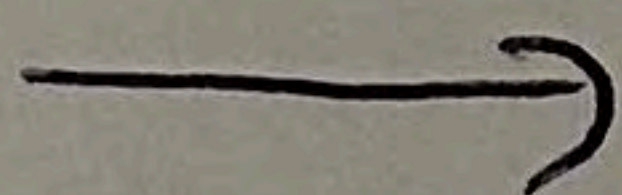
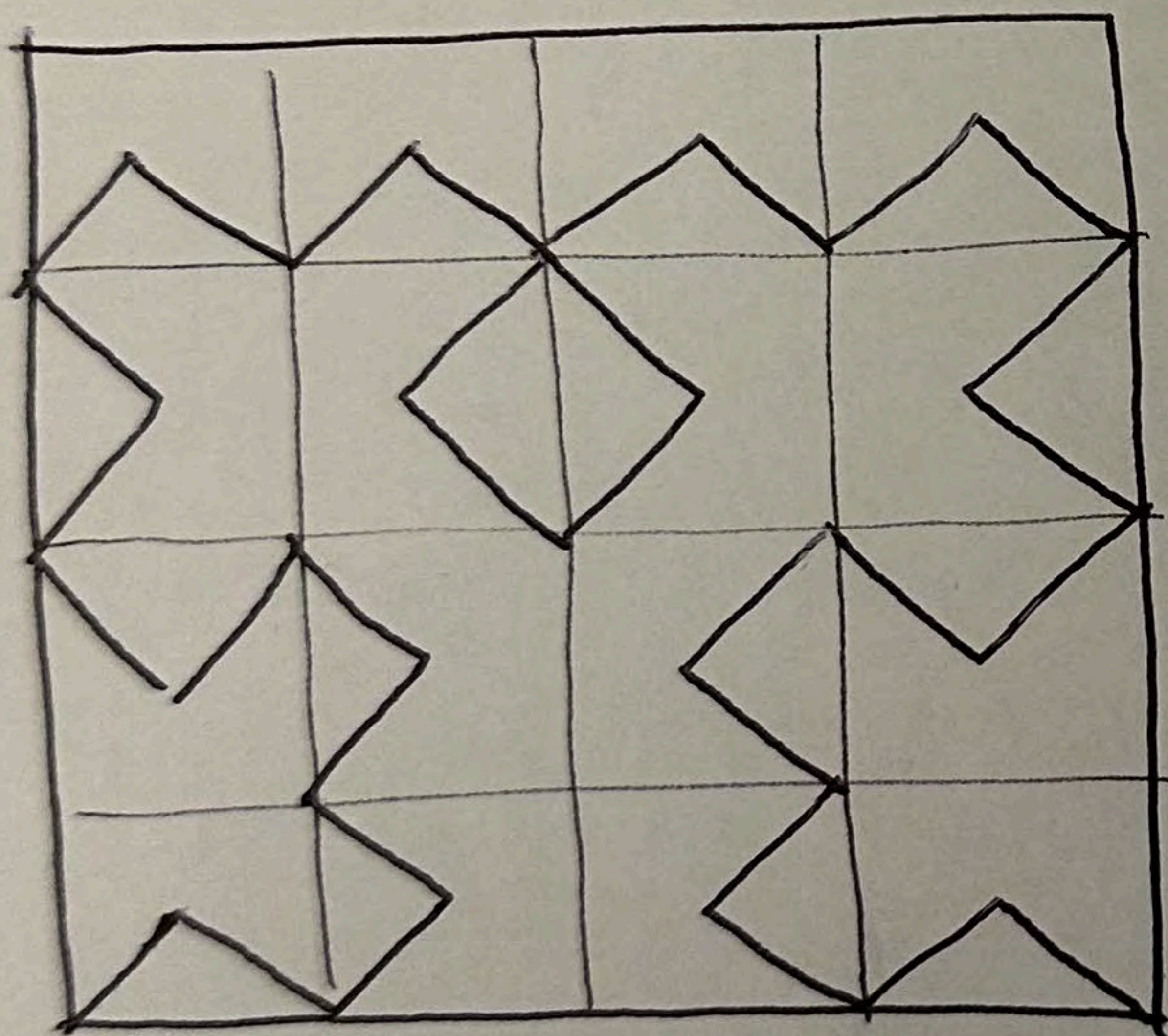
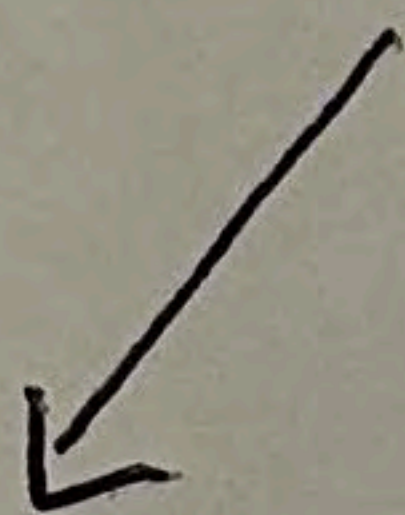
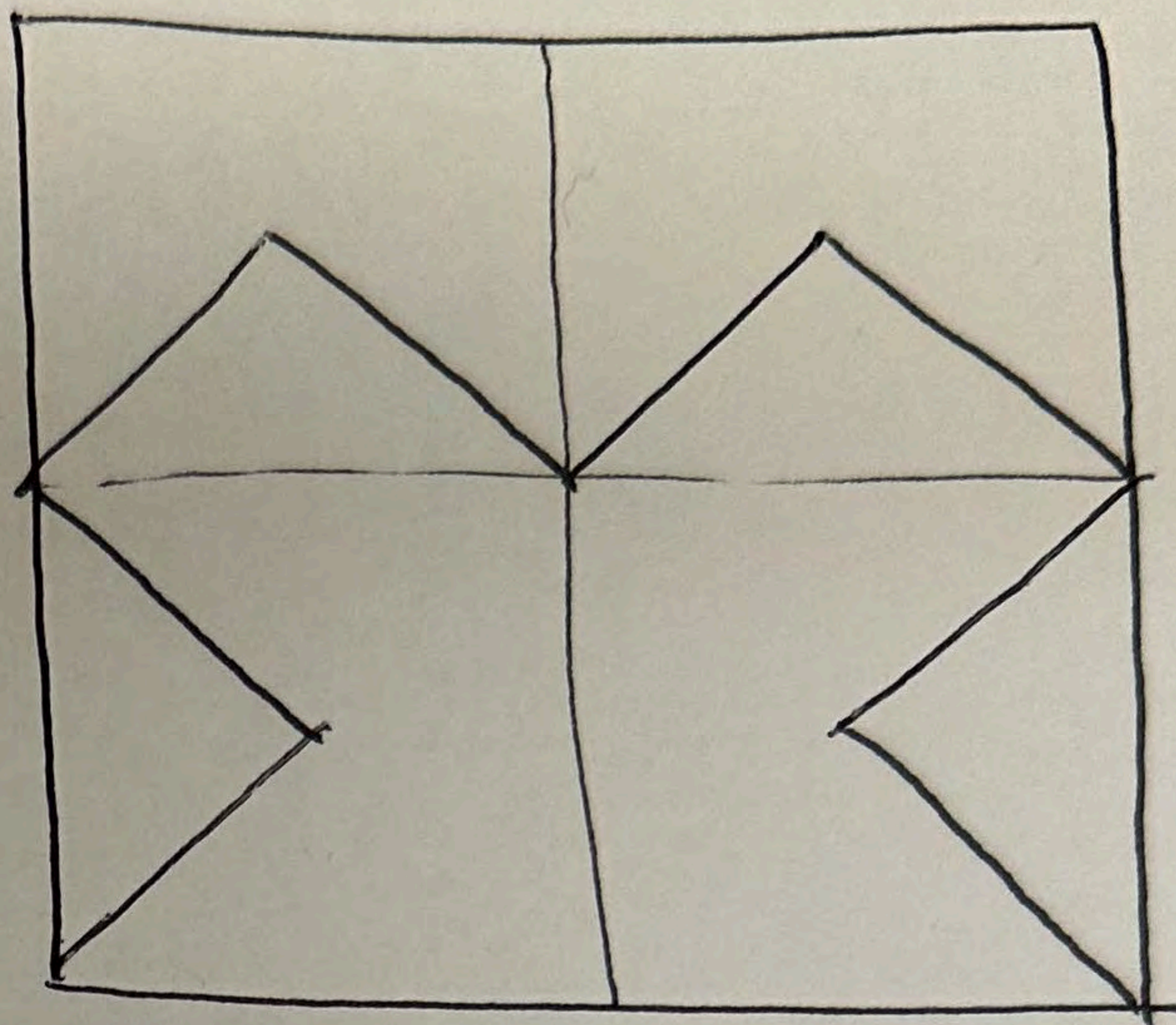
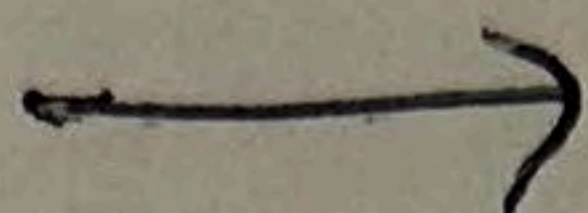
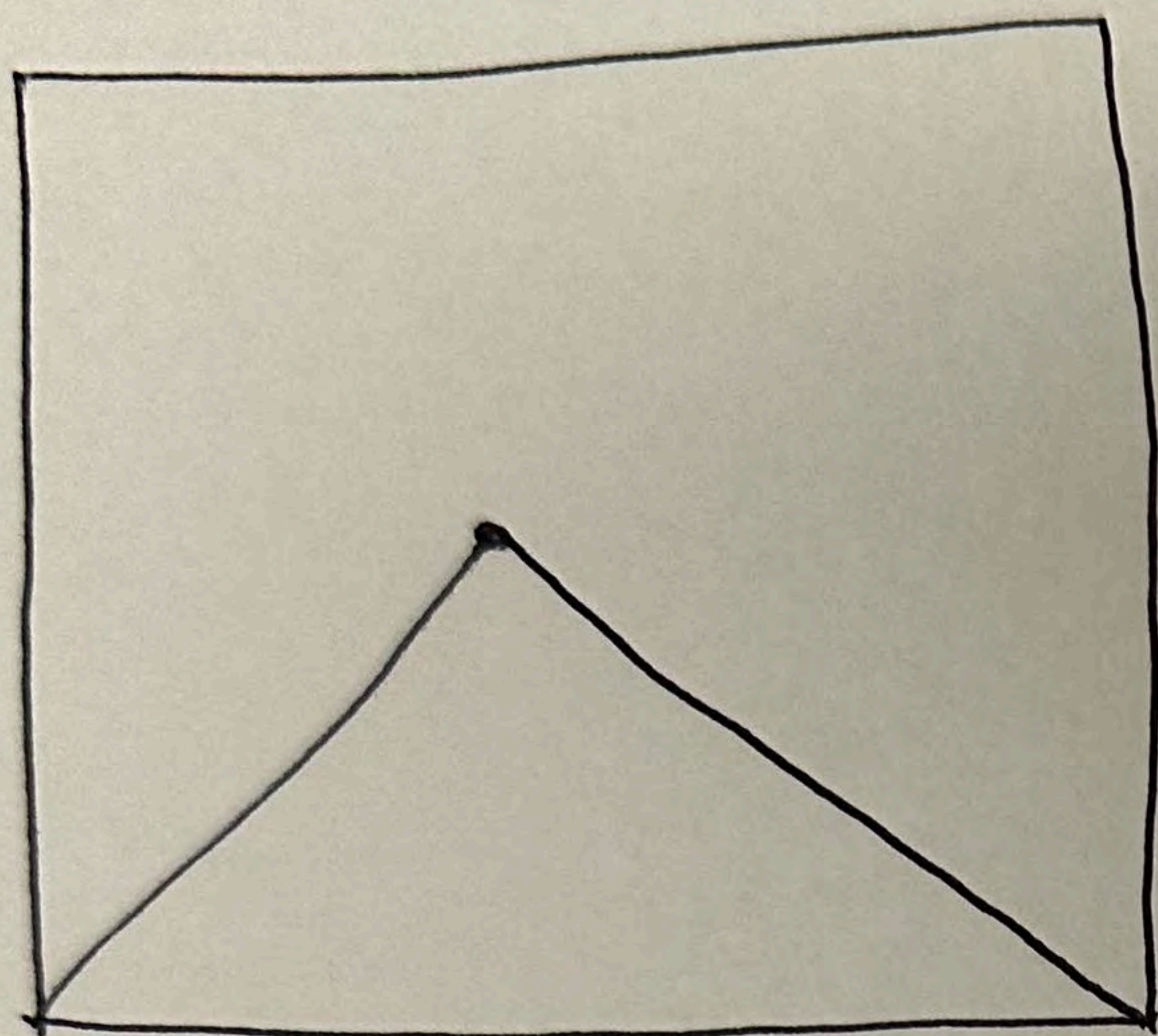
cf. n -point metric spaces embed into ℓ_∞^{n-1} .

separable metric spaces embed into ℓ_∞ .

§44. A Space-Filling Curve by Peano.

Thm. There is a continuous surjection $[0,1] \rightarrow [0,1]^2$.

pf)



...

MA 109A: LECTURE NOTES FOR SECTION 45

1. ASCOLI'S THEOREM

The Heine-Borel theorem, which you have learned in previous courses, states that the subsets of $\mathbb{R}^n$ which are compact are precisely the closed and bounded subsets. It's not hard to see that it characterizes the precompact¹ subsets of $\mathbb{R}^n$ as the bounded subsets. In general, a compact subspace of a metric space will be closed and bounded, and a precompact subspace will be bounded, but the converse need not hold, as the following example illustrates.

Example 0.1. Consider the metric space $\mathcal{C}([0, 1], \mathbb{R})$ with the sup norm ρ . The closed and bounded subset $\{f \in \mathcal{C}([0, 1], \mathbb{R}) : \rho(0, f) = \|f\|_{\text{sup}} \leq 1\}$ is not compact, as the sequence $f_n(x) = x^n$ does not admit a convergent subsequence.

So, to obtain compactness, closedness and boundedness will in general not be enough. We need extra conditions depending on the ambient metric space in question. Section 45 is devoted to characterizing which subsets of $\mathcal{C}(X, \mathbb{R}^n)$ are compact or precompact, when X is a compact topological space. More precisely, we prove Ascoli's theorem²:

Theorem 1 (Ascoli's theorem, Corollary 45.5). *Let X be a compact topological space, and endow $\mathcal{C}(X, \mathbb{R}^n)$ with the sup norm. Then a subfamily $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ is compact if and only if $\mathcal{F}$ is closed, sup norm-bounded, and equicontinuous. A subfamily $\mathcal{F}$ of $\mathcal{C}(X, \mathbb{R}^n)$ has compact closure if and only if $\mathcal{F}$ is sup norm-bounded and is equicontinuous.*

How would one accomplish such a feat? More importantly, how would one even define equicontinuity? Recall earlier in the course that we proved the following theorem.

Theorem 2. *Let Y be a metric space. The following are equivalent.*

- Y is compact.
- Y is limit point compact.
- Y is sequentially compact.
- Y is complete and totally bounded.

Here, to refresh your memory, “totally bounded” means the following.

Definition 3. *A metric space (Y, d) is said to be totally bounded if for any $\varepsilon > 0$ there are finitely many points $y_1, \dots, y_n \in Y$ such that $Y = B(y_1, \varepsilon) \cup \dots \cup B(y_n, \varepsilon)$.*

¹If you recall from a previous problem set, this means that the closure is compact.

²Historical note: according to Wikipedia, Ascoli proved the sufficiency in between 1883 and 1884, and Arzelà proved the necessity and polished the results in 1895. I usually hear it being called the “Arzelà-Ascoli theorem”, but since the book decides to call it the “Ascoli theorem,” let's follow the notation of the book.

Note that if Y is a complete ambient space, its closed subspaces are complete, so its compact subspaces are precisely its closed totally bounded subspaces.

What does “totally bounded” mean in the function space $\mathcal{C}(X, \mathbb{R}^d)$? It should first imply bounded, since totally bounded subsets are bounded (as you can readily check). It also implies the aforementioned *equicontinuity*, which we now define.

Definition 4. Let X be a topological space and (Y, d) a metric space. A family $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is said to be equicontinuous if for every $x_0 \in X$ and $\delta > 0$ there is an open neighborhood U of x_0 such that

$$\forall x \in U \quad \forall f \in \mathcal{F} \quad d(f(x), f(x_0)) < \delta.$$

Note that a single continuous function $f \in (X, Y)$ automatically forms an equicontinuous family $\{f\}$, and that a finite collection of continuous functions $f_1, \dots, f_n \in (X, Y)$ form an equicontinuous family $\{f_1, \dots, f_n\}$.

2. PROOF OF ASCOLI’S THEOREM

Let’s begin by seeing that if a family $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is totally bounded, then it is equicontinuous.

Lemma 5 (Lemma 45.2). Let X be a topological space and let (Y, d) be a metric space. Endow $\mathcal{C}(X, Y)$ with the uniform metric³ $\bar{\rho}$. Then a totally bounded family $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is equicontinuous.

Proof. Let $x_0 \in X$ and $\delta \in (0, 1)$. Since $\mathcal{F}$ is totally bounded, there are functions $f_1, \dots, f_n \in \mathcal{F}$ such that $\mathcal{F} \subseteq \bigcup_{i=1}^n B_{\bar{\rho}}(f_i, \delta/3)$. Since the family $\{f_1, \dots, f_n\}$ is equicontinuous, there is an open set $U \ni x_0$ such that

$$\forall x \in U \quad \forall i = 1, \dots, n \quad d(f_i(x), f_i(x_0)) < \delta/3.$$

A routine triangle inequality now shows that for any $x \in U$ and any $f \in \mathcal{F}$, choosing $i \in \{1, \dots, n\}$ such that $f \in B_{\bar{\rho}}(f_i, \delta/3)$,

$$d(f(x), f(x_0)) \leq d(f(x), f_i(x)) + d(f_i(x), f_i(x_0)) + d(f_i(x_0), f(x_0)) < \delta.$$

□

Conversely, let’s see that if a family $\mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R}^d)$ is bounded and equicontinuous, then it is totally bounded. We will prove the slightly more general version of this statement given as follows. (The below is more general because if $\mathcal{F}$ is bounded, say $|f| \leq M$ for all $f \in \mathcal{F}$, then we take $Y = \overline{B_M} \subseteq \mathbb{R}^d$).

Lemma 6 (Lemma 45.3). Let X be a compact topological space and let (Y, d) be a compact metric space. Endow $\mathcal{C}(X, Y)$ with the sup metric ρ . Then an equicontinuous family $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is totally bounded.

³To refresh your memory, the sup and uniform metrics are

$$\rho(f, g) = \sup_{x \in X} d(f(x), g(x)), \quad \bar{\rho}(f, g) = \sup_{x \in X} \min\{d(f(x), g(x)), 1\}.$$

In general, the uniform metric is well defined over the continuous functions, and if X is in addition compact, then the sup norm is defined and satisfies $\bar{\rho} = \min\{\rho, 1\}$. Since we are not assuming that X is compact in this lemma, we are working with the uniform metric.

Proof. Let $\delta > 0$. Since Y is compact, we may find points $y_1, \dots, y_n \in Y$ such that $Y = \bigcup_{i=1}^n B_d(y_i, \delta/3)$. On the other hand, for each $x \in X$ we may find a $\delta/3$ -equicontinuous neighborhood U_x ; cover X by $\delta/3$ -equicontinuous neighborhoods $U_1, \dots, U_m$ of points $x_1, \dots, x_m$. For each function $\alpha : \{x_1, \dots, x_m\} \rightarrow \{y_1, \dots, y_n\}$, denote

$$\mathcal{F}_\alpha = \{f \in \mathcal{F} : f(x_j) \in B_d(y_{\alpha(j)}, \delta/3) \forall j = 1, \dots, m\}.$$

It is clear that

$$\mathcal{F} = \bigcup_{\alpha : \{x_1, \dots, x_m\} \rightarrow \{y_1, \dots, y_n\}} \mathcal{F}_\alpha.$$

Since this is a finite union, it is enough to see that each $\mathcal{F}_\alpha$ is contained in a sup ball of radius δ . It may happen that some $\mathcal{F}_\alpha$ are empty, in which case there is nothing to prove; otherwise, choosing $f_\alpha \in \mathcal{F}_\alpha$, we see that $f \in B_\rho(f_\alpha, \delta)$: indeed, for any $x \in X$ we have $x \in U_j$ for some $j = 1, \dots, m$, and since $f(x_j), f_\alpha(x_j) \in B_d(y_{\alpha(j)}, \delta/3)$, we have

$$d(f(x), f_\alpha(x)) \leq d(f(x), f(x_j)) + d(f(x_j), f_\alpha(x_j)) + d(f_\alpha(x_j), f_\alpha(x)) < \delta.$$

□

We are now ready to prove Ascoli's theorem.

Proof of Theorem 1. If $\mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R}^d)$ is compact, then it is closed, (sup norm-)bounded, and totally bounded, and hence by Lemma 5 is equicontinuous. Conversely, if $\mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R}^d)$ is closed, sup norm-bounded and equicontinuous, then it is complete (being a closed subspace of a complete space) and totally bounded (by Lemma 6, being sup norm-bounded and equicontinuous), and hence is compact.

Now $\mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R}^d)$ is precompact iff $\overline{\mathcal{F}} \subseteq \mathcal{C}(X, \mathbb{R}^d)$ is compact iff $\overline{\mathcal{F}}$ is closed, sup norm-bounded and equicontinuous iff $\mathcal{F}$ is sup norm-bounded and equicontinuous. (The last equivalence requires some manual checking.) □

3. POINTWISE BOUNDEDNESS

In section 47, we're going to formulate Ascoli's theorem in the more general setting when X need not be compact. In that setting, we will work with the topology of *compact convergence* (to be defined in section 46), which you may better know as “uniform convergence on compact sets.” As such, we will need yet another replacement for the notion of sup norm-boundedness; that will be of *pointwise boundedness*.

Definition 7. If (Y, d) is a metric space, a subfamily $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ is said to be pointwise bounded if for each $a \in X$, the set

$$\mathcal{F}_a := \{f(a) : f \in \mathcal{F}\}$$

is a bounded subset of Y .

The connection to sup norm-boundedness is as follows.

Lemma 8. Let X be a compact topological space, let (Y, d) be a metric space, and let $\mathcal{F} \subseteq \mathcal{C}(X, Y)$ be pointwise bounded and equicontinuous. Then $\mathcal{F}$ is sup norm-bounded.

Proof. Cover X by finitely many 1-equicontinuous neighborhoods $U_1, \dots, U_m$ of points $x_1, \dots, x_m \in X$. Then for any $f \in \mathcal{F}$ and $x \in X$, $f(x)$ is contained in the 1-neighborhood of the bounded set $F_{x_1} \cup \dots \cup F_{x_m} \subseteq Y$. $\square$

This lemma, together with Theorem 1, implies the “classical” version of Ascoli’s theorem.

Theorem 9 (Classical version of Ascoli’s theorem, Theorem 45.4). *Let X be a compact topological space and endow $\mathcal{C}(X, \mathbb{R}^d)$ with the uniform topology. Then $\mathcal{F} \subseteq \mathcal{C}(X, \mathbb{R}^d)$ is precompact iff $\mathcal{F}$ is equicontinuous and pointwise bounded.*

See Theorem 47.1 if you are curious as to what the general version would be for noncompact X .

MA 109A: LECTURE NOTES FOR CHAPTER 8

1. SECTION 48: BAIRE SPACES

Motivating question:

Question 1.

- (1) Can the plane $\mathbb{R}^2$ be expressed as a countable union of lines?
- (2) Can the sphere $\mathbb{S}^2$ be expressed as a countable union of circles?

If you know measure theory, you can answer “no” to both questions, arguing that lines and circles have zero area and hence their countable unions also have zero area and cannot be the entire 2-dimensional space. Let’s tie one hand behind our back¹ and try to prove the impossibility using purely topological arguments.

Indeed, in some sense lines lack “fullness” in $\mathbb{R}^2$. Can we express this without the concept of area? We can observe that lines have *empty interior* in $\mathbb{R}^2$. This motivates the following definition.

Definition 2. Let X be a topological space and let $S \subseteq X$ be a subset.

- (1) Let $O \subseteq X$ be an open set. S is not dense in O if $O \cap \overline{S}$ is a proper subset of O .
- (2) S is nowhere dense if S is not dense in any nonempty open set in X . Equivalently, $\overline{S}$ has empty interior.
- (3) S is of the first category, if S is the union of a countable family of sets S_i , each of which is nowhere dense. Equivalently, S is contained in the union of a countable collection of closed sets of X with empty interior.
- (4) S is of the second category if it is not of the first category.
- (5) X is a Baire space if every nonempty open set in X is of the second category.

In our example above, lines and circles are nowhere dense in $\mathbb{R}^2$ and $\mathbb{S}^2$, respectively. A countable union of lines and circles would then be sets of the first category, and our intuition is that $\mathbb{R}^2$ and $\mathbb{S}^2$ are of the second category. The main theorem of this section is the following.

Theorem 3 (Baire category theorem). *Compact Hausdorff spaces and complete metric spaces are Baire spaces.*

The Baire category theorem states that $\mathbb{R}^2$ and $\mathbb{S}^2$ are Baire spaces, meaning that you cannot cover any open set, however small, in $\mathbb{R}^2$ (resp. $\mathbb{S}^2$) with countably many lines (resp. circles).

Another application of the Baire category theorem is that pointwise limits of continuous functions have dense continuity points.

¹I’m not trying to give the impression that category arguments replace measure theory arguments. Rather, they are complementary viewpoints on the idea of “density.” If you want to know more about their interaction, see *Measure and Category* by John Oxtoby (admittedly I haven’t read this book, I just often saw it rolling around in math departments), where you can find many more applications of category-type arguments.

Theorem 4. *Let X be a Baire space, let Y be a metric space, and let $f_n : X \rightarrow Y$ be a sequence of continuous functions converging pointwise to $f : X \rightarrow Y$. Then the set of points at which f is continuous is dense in X .*

Proof. See Lemma 48.4 and Theorem 48.5 of Munkres. $\square$

Munkres gives another application of the Baire category theorem by constructing a nowhere differentiable function (Section 49).

The Baire category theorem has importance beyond the above examples. Besides the applications you can find in the book *Measure and Category* mentioned in the footnote, the Baire theorem is foundational in proving basic results about functional analysis (which you might have guessed that I am quite fond of). I might give some applications to functional analysis in this week's problem set.

I owe you the proof of the Baire category theorem. As a preparation, let's prove Cantor's intersection theorem for complete metric spaces.

Lemma 5 (Cantor's intersection theorem for complete metric spaces). *Let X be a complete metric space, and let $C_1 \supset C_2 \supset \cdots$ be a nested sequence of nonempty closed sets with $\text{diam } C_n \rightarrow 0$ as $n \rightarrow \infty$. Then $\bigcap_n C_n$ consists of exactly one point.*

Proof. Because $\text{diam}(\bigcap_n C_n) \leq \text{diam } C_n \rightarrow 0$ as $n \rightarrow \infty$, we have $\text{diam}(\bigcap_n C_n) \leq 0$. So there is at most one point in the intersection. To prove that the intersection is nonempty, choose $x_n \in C_n$. Then $x_n, x_m \in C_{\min\{n,m\}}$ so that $d(x_n, x_m) \leq \text{diam } C_{\min\{n,m\}}$ and hence it follows that $\{x_n\}_{n=1}^\infty$ is a Cauchy sequence. By completeness of X , there is a limit $x_n \rightarrow x$ as $n \rightarrow \infty$. Because each C_n is closed, and $x_m \in C_n$ for $m \geq n$, we have $x \in C_n$ for all n , proving $x \in \bigcap_n C_n$. $\square$

Let's also prove the following preparatory lemma.

Lemma 6.

- X is a Baire space if and only for any countable collection $\{F_n\}$ of closed sets of X having empty interiors, their union $\bigcup_n F_n$ also has empty interior in X .
- X is a Baire space if and only if for any countable collection $\{O_n\}$ of dense open sets of X , their intersection $\bigcap_n O_n$ is also dense in X .

Proof. The second assertion follows from the first by taking complements. For the first assertion, by definition X is a Baire space iff every nonempty open set is of the second category, which means that no nonempty open set can be contained in the union of closed sets with empty interior. This is equivalent to the stated property. $\square$

We are now ready to prove the Baire category theorem.

Proof of the Baire category theorem. By Lemma 6, it is enough to show that for any countable collection $\{F_n\}$ of closed sets of X with empty interiors, their union $\bigcup_n F_n$ also has empty interior. So given any nonempty open set U_0 of X , it is enough to find a point x of U_0 not lying in any A_n .

By hypothesis, A_1 cannot contain U_0 , so we may choose $y \in U_0 \setminus A_1$. Since U_0 is closed and A_1 is closed, we may invoke regularity of X to choose a neighborhood U_1 of y such that

$$\overline{U_1} \cap A_1 = \emptyset, \quad \overline{U_1} \subseteq U_0,$$

with the additional property that $\text{diam } U_1 \leq 1$ if X is a metric space.

Inductively, given the open set U_{n-1} , we repeat the same procedure to choose a nonempty open set U_n such that

$$\overline{U_n} \cap A_n = \emptyset, \quad \overline{U_n} \subseteq U_{n-1},$$

with the additional property that $\text{diam } U_n \leq 1/n$ if X is a metric space.

We observe that $\bigcap_n \overline{U_n}$ is nonempty. Indeed, if X is compact Hausdorff, then $\overline{U_1} \supset \overline{U_2} \supset \cdots$ is a descending sequence of nonempty compact sets, and we may invoke the definition of compactness in terms of the finite intersection property. If X is complete metric, this is because of the above Cantor intersection theorem (Lemma 5).

Now choose $x \in \bigcap_n \overline{U_n}$. Then $x \in U_0$ by the nested property, while $x \notin A_n$ since $\overline{U_n} \cap A_n = \emptyset$. $\square$

2. SECTION 50: INTRODUCTION TO DIMENSION THEORY

I will go briefly over this section² because I want to go on to some algebraic topology for this course.

Definition 7.

- (1) A collection $\mathcal{A}$ of subsets of the space X has order at most m if no point of X lies in more than m elements of $\mathcal{A}$.
- (2) A topological space X is said to have topological dimension at most m if every open covering of X has an open refinement of order at most $m + 1$.

The main theorem of this section is the following.

Theorem 8. *Every m -manifold has topological dimension at most m .*

So, for example, given any open covering of an m -manifold, you can construct a partition of unity, only at most $m + 1$ of which are nonzero near each point. You can guess that having such quantitative bounds allows you to construct nice functions on manifolds, such as embeddings:

Theorem 9. *Every m -manifold can be embedded in $\mathbb{R}^{2m+1}$ as a closed subspace.*

I think that's enough analysis-y topology for the term. Let's mark the end of part 1 of Munkres and move on to the algebraic methods of part 2.

²Interested readers are referred to *Dimension Theory* by Hurewicz and Wallman (which, again, admittedly I have not read).